

**Problem 1**

The scheme in the left figure works as an integrator. OA has  $A_0 = 10^5$  and  $GBWP = 1$  MHz; component values are  $R_1 = R_2 = 10$  k $\Omega$ ,  $C = 10$  nF.

1. Compute the ideal closed-loop gain of the stage.
2. Compute the closed-loop gain of the stage (i.e., accounting for the finite  $A(s)$ ).
3. Compute the total output rms noise due to the input voltage noise of the OAs,  $\sqrt{S_V} = 10$  nV/ $\sqrt{\text{Hz}}$ .
4. Propose a simple modification to the circuit to achieve an approximate integrator, i.e., having a pole located at a low, but non-zero, frequency.

**Problem 2**

A rectangular signal of amplitude  $A \approx 50$   $\mu\text{V}$  and duration  $T_p = 1$   $\mu\text{s}$  is hidden amid a white noise having a bilateral PSD  $\lambda = 10^{-14}$   $\text{V}^2/\text{Hz}$ .

1. A gated integrator is used to detect the pulse amplitude. Find the resulting  $S/N$  and specify the output sampling time.
2. Draw (on *quoted* diagrams) the output signal and the autocorrelation of the output noise.
3. Now consider the signal and noise computed in #2.2. Find the optimum filter for this condition and the corresponding  $S/N$ . Discuss the result.
4. A digital filter is used in place of the GI in #2.1. Sampling is not ideal, but is approximated by an LPF with time constant  $t_a = 1$  ns (Fig. on the right). Find the resulting  $S/N$ .

**Question**

Describe the temperature compensation techniques for a Wheatstone bridge.

For a correct evaluation, you are asked to write your answers in a readable way; thank you

Do a good job!

# Solution

## Problem 1

### 1.1

The KCL at the non-inverting input of the OA reads

$$\frac{V_o - V^+}{R_2} = sCV^+ + \frac{V^+ - V_i}{R_1},$$

where the voltage  $V^+$  at the non-inverting input of the OA (equal to  $V^-$ ) can be straightforwardly written as

$$V^+ = V^- = V_o \frac{R_1}{R_1 + R_2}.$$

With simple substitution we get

$$V_o = \frac{R_1 + R_2}{R_1} \frac{V_i}{sCR_1},$$

i.e., a non-inverting integrator. The scheme is actually based on a Howland current-source topology with a capacitor as load, and is basically the same that can be found in the 7/25/08 exam. A variation on the theme, with unbalanced resistors, is in the 7/23/14 exam: there's nothing new under the sun!

### 1.2

We need to compute the open-loop gain, either directly or via calculation of  $G_{loop}$ . In the first case, we disconnect the feedback at the OA output, connecting both  $R_2$  resistors to ground. The transfer becomes then

$$G_{OL} = A(s) \frac{R_2 \parallel 1/sC}{R_1 + R_2 \parallel 1/sC} = A(s) \frac{R_2}{R_1 + R_2} \frac{1}{1 + sC(R_1 \parallel R_2)},$$

which is a transfer function with two poles, at 10 Hz and at  $1/\pi CR_1 \approx 3.2$  kHz. These functions are shown in Fig. 1 (left), together with the closed-loop gain, which is the minimum of the two. The two poles in  $G$  (see markers in Fig. 1, left) can be computed as:

$$\begin{aligned} \frac{R_1 + R_2}{R_1} \frac{1}{sCR_1} = A_0 \frac{R_2}{R_1 + R_2} &\Rightarrow f_L = \frac{2}{\pi A_0 CR_1} = 6.4 \text{ mHz} \\ \frac{R_1 + R_2}{R_1} \frac{1}{sCR_1} = A_0 \frac{R_2}{R_1 + R_2} \frac{2}{s^2 \tau CR_1} &\Rightarrow f_H = \frac{A_0}{4\pi\tau} = 500 \text{ kHz}. \end{aligned}$$

Of course, the same results are obtained via  $|G_{loop}| = 1$ , where  $G_{loop} = -A(s)sCR/(4 + 2sCR)$  (no stability problems).

### 1.3

Grounding the input and turning on the voltage noise source  $V_n$ , we can easily write expressions for  $V^+$  and  $V^-$  and equal them:

$$V_o \frac{R_1 \parallel 1/sC}{R_2 + R_1 \parallel 1/sC} + V_n = V_o \frac{R_1}{R_1 + R_2} \Rightarrow V_o = -V_n \frac{R_1 + R_2}{R_1} \frac{1 + sC(R_1 \parallel R_2)}{sC(R_1 \parallel R_2)},$$

which gives a noise contribution diverging for both low and high frequencies. We should therefore account for the limitations discussed previously: a HF pole at  $f_H = 500$  kHz and a LF pole at  $f_L = 6.4$  mHz, where the gain flattens at  $A_0$ . We have therefore:

$$\overline{V_o^2} \approx S_V \left( A_0^2 \frac{\pi}{2} f_L + 4 \frac{\pi}{2} f_H \right) \approx 10^{-8} \text{ V}^2,$$

meaning that the final rms value is about 100  $\mu\text{V}$ .

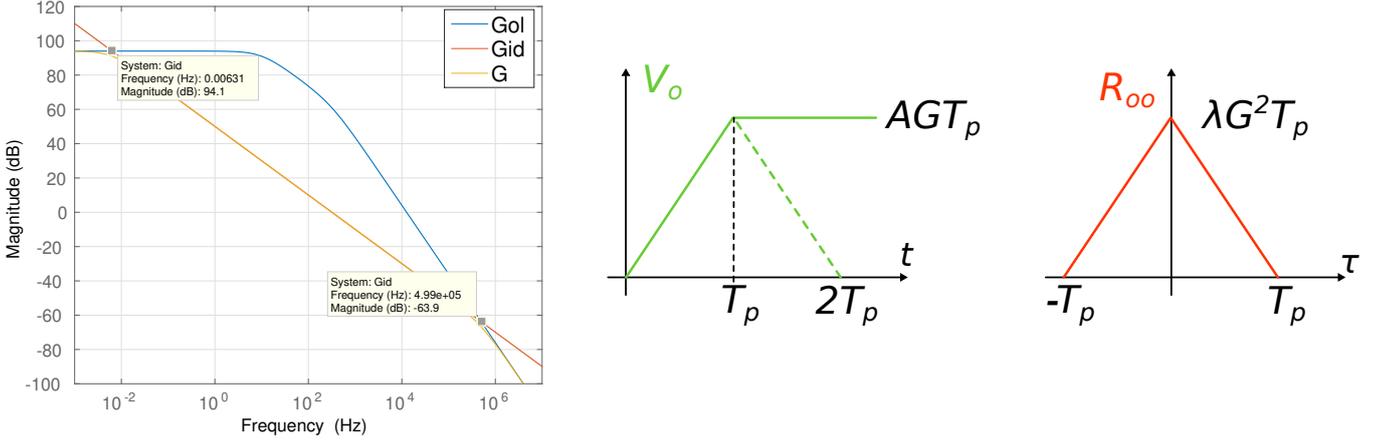


Figure 1: Left: Bode magnitude diagram for  $G_{OL}$ ,  $G_{id}$  and  $G$ . Right: noise autocorrelation and output signal from the GI.

### 1.4

By looking at the expression for the ideal gain, it is clear that the problem can be solved by placing a resistor  $R_C$  in parallel to the capacitor  $C$ , obtaining:

$$\frac{V_o}{V_i} = \frac{R_1 + R_2}{R_1} \frac{Z}{R_1} = \frac{R_1 + R_2}{R_1} \frac{R_C}{R_1} \frac{1}{1 + sCR_C}$$

A slightly less obvious approach is to unbalance the two branches of the feedback loop, e.g., by adding a suitable resistor in series to the upper  $R_1$ . It is easy to see that if we put  $V^- = (0.5 + x)V_o$  we obtain

$$\frac{V_o}{V_i} = \frac{1}{2x + sCR(0.5 + x)}$$

i.e., an approximate integration. The same result is achieved by increasing  $R_2$  in the lower branch.

## Problem 2

### 2.1

Given the shape of the signal, it is wise to integrate over the whole pulse duration, getting for the output  $S/N$ :

$$\frac{S}{N} = \frac{GAT_p}{\sqrt{G^2\lambda T_p}} = A\sqrt{\frac{T_p}{\lambda}} = 0.5.$$

where  $G$  is the GI stage gain. The output signal is taken at time  $t = T_p$ .

### 2.2

We know that the signal output from a time-varying filter can be expressed as

$$V_o(t) = \int_0^t V_i(\tau)w(t, \tau)d\tau = AGt \quad \forall t < T_p,$$

where  $w$  is the filter weighting function. In our case  $V_i$  and  $w$  are rectangular functions and the result is the ramp signal shown in Fig. 1 (right). After reaching the integration time, the signal is held constant and eventually dropped to begin a new operation.

Had we assumed an LTI integrator with an integration window of duration  $T_p$ , we would have obtained a triangular signal (dashed line). However, even if the final value is the same, this is not how a GI actually works.

As for the output noise, its autocorrelation is given by

$$R_{oo}(t_1, t_2) = \int R_{nn}(\tau)k_{w_{12}}(\tau)d\tau = \lambda k_{w_{12}}(0) = \lambda \int w(t_1, \alpha)w(t_2, \alpha)d\alpha,$$

which is again proportional to the autocorrelation of the rectangular weighting function  $w$ . The result is

$$R_{oo}(\tau) = \lambda G^2 T_p \text{tri}(\tau, T_p),$$

also shown in Fig. 1b (right).

### 2.3

The GI is already the optimum filter for the initial problem, so nothing we can do at its output can further increase the signal to noise ratio! That's what "optimum" means, after all, isn't it?

To convert the skeptics, let's compute the optimum filter for the case under exam, keeping in mind that the noise is *not* white (see Fig. 1, right): in the frequency domain, the Fourier transform of the optimum filter weighting function is given by

$$W(f) = \frac{V_o(f)}{S_{no}(f)} = \frac{AGT_p^2 \text{sinc}^2(\pi f T_p) e^{-j2\pi f T_p}}{\lambda G^2 T_p^2 \text{sinc}^2(\pi f T_p)} = K e^{-j2\pi f T_p},$$

whose antitransform is

$$w(t, \tau) = \delta(\tau - T_p).$$

This means that the best we can do is just sample the output signal at its peak, which is exactly what the optimum filter does.

### 2.4

We start by recalling that a GI is approximated in the digital realm by a *uniform* sampling, meaning that the increase in  $S/N$  will follow a  $\sqrt{N}$  law, provided noise samples are uncorrelated. The value of  $S/N$  after each sampling event is given by the usual expression for the LPF:

$$\left(\frac{S}{N}\right) = \frac{A}{\sqrt{\lambda/2t_a}} = A\sqrt{\frac{2t_a}{\lambda}},$$

so that we now just need to find the maximum value of  $N$ : to correctly sample the input signal, we need  $T_s \approx 5t_a$ , so the maximum number of samples is given by

$$N = \frac{T_p}{5t_a},$$

meaning that

$$\left(\frac{S}{N}\right)_{max} = A\sqrt{\frac{2t_a}{\lambda}}\sqrt{N} = A\sqrt{\frac{2T_p}{5\lambda}} = 0.32,$$

of course smaller than what obtained in #2.1. Note that noise samples are taken at an interval  $T_s = 5t_a$  while the noise correlation time is  $t_n \approx 4t_a$ ; the non-correlation hypothesis is at its limit.