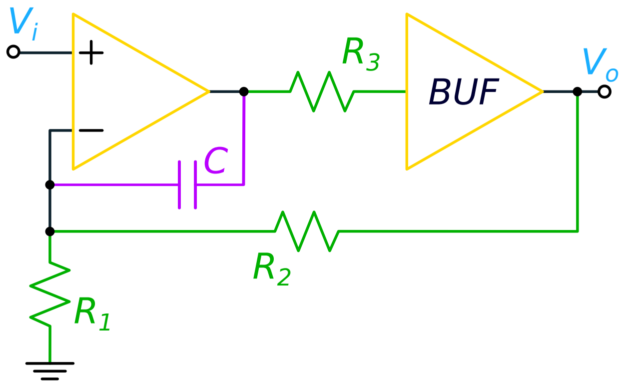
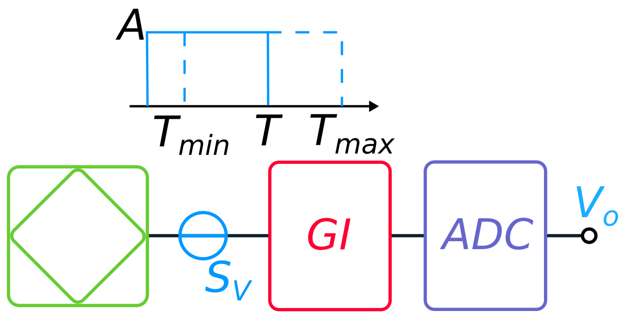
For a correct evaluation, please write your answers in a readable way; thank you




Solving six points correctly gives you 30/30

## Problem 1

The scheme in the left figure is a line driver for audio applications. Parameter values are $R_1 = 1$ kΩ, $R_2 = 4$ kΩ, $R_3 = 100$ Ω, $C = 39$ pF. The OA has $A_0 = 120$ dB and $GBWP = 15$ MHz. The (ideal) buffer stage has unity gain.

1. Evaluate the ideal gain (be careful!).
2. Evaluate the loop gain and discuss the stability.
3. Compute the output noise voltage PSD due to the the equivalent voltage noise sources of the amplifiers, $\sqrt{S_V} = 4\ \mathrm{nV}/\sqrt{\mathrm{Hz}}$.
4. Consider a unit step voltage at the input. What is the minimum value of the OA slew rate that grants a linear response?

## Problem 2

A sensor outputs rectangular signals of amplitude $A = 1$ mV and unknown width $T$ ranging from $T_{min} = 10$ ns to $T_{max} = 1\ \mu$s, on top of a white noise with unilateral PSD $\sqrt{S_V} = 20\ \mathrm{nV}/\sqrt{\mathrm{Hz}}$. A gated integrator is used to measure $T$, followed by an ADC with input range between 0 and 5 V, as shown in the right figure.

1. Set the integration time of the GI and evaluate the minimum detectable value of $T$.
2. Find the GI gain and the minimum number of ADC bits to ensure a correct detection.
3. A flicker noise with corner frequency $f_{nc} = 50$ kHz is present at the GI input. Find the expression of $S/N$ when an HPF with time constant $T_F$ is placed before the GI.
4. Find the optimum value of $T_F$ for the case of flicker noise only (consider that the equation $\log(x/\pi) = -1/(2x)$ has approximate solution $x = 0.17$).

Allowed time: 2 hours 45 minutes – Do a good job!

Results will be posted by January 23rd

Mark registration: Wednesday, January 29th

# Solution

## Problem 1

### 1.1

At first, we know that the buffer stage has a high input impedance (is a voltage amplifier), so no current flows in $R_3$ that can be neglected, and the voltage $V_o$ is also present at the OA output.
Now, as the differential input voltage of the OA is zero, the voltage at the $R_1 - R_2$ midpoint is $V_i$, and the current in $R_1$ flows via $C$ and $R_2$, that behave as if they are in parallel! The expression for the gain becomes then

$$\frac{V_o}{V_i} = 1 + \frac{R_2 \parallel 1/sC}{R_1} = \frac{R_1 + R_2}{R_1}\frac{1 + sC\, R_1 \parallel R_2}{1 + sCR_2}.$$

The amplifier has a gain of 5 up to a bandwidth $f_p = 1/(2\pi CR_2) = 1$ MHz, that drops to 1 beyond $f_z = 1/(2\pi C\, R_1 \parallel R_2) = 5$ MHz.

### 1.2

We break the loop at the OA output and apply a test voltage $V_T$. Recalling what said before, we can write:

$$V^- = V_T\frac{R_1}{R_1 + R_2 \parallel 1/sC},$$

from which

$$G_{loop} = -A(s)\frac{R_1}{R_1 + R_2 \parallel 1/sC} = -A(s)\frac{R_1}{R_1 + R_2}\frac{1 + sCR_2}{1 + sC\, R_1 \parallel R_2}.$$

Pole and zero positions are 1 and 5 MHz, respectively, beyond which $G_{loop}$ follows $-A(s)$ and the system remains stable with phase margin of 90° and $f_{0dB} = GBWP = 15$ MHz. As a side comment, it is interesting to note that the circuit remains stable even if the pole added by the buffer bandwidth is considered, thanks to $C$. Calculations are in the Appendix..

### 1.3

The voltage noise source of the OA is subjected to the same transfer as the input signal. As for the noise of the buffer, we can refer to the scheme in Fig. 1: the voltage at the OA output is $V_o - V_n$, and the OA inverting input is grounded, meaning that no current flows in $R_1$. The voltage divider between $R_2$ and $C$ leds then to (linear superposition):

$$V^- = V_o\frac{1}{1 + sCR_2} + (V_o - V_n)\frac{sCR_2}{1 + sCR_2} = 0 \Rightarrow V_o = V_n\frac{sCR_2}{1 + sCR_2},$$

from which we obtain the final expression:

$$V_o = V_n^{OA}\frac{R_1 + R_2}{R_1}\frac{1 + sCR_1 \parallel R_2}{1 + sCR_2} + V_n^{BUF}\frac{sCR_2}{1 + sCR_2}$$

$$S_{V_o} = S_V^{OA}\left(\frac{R_1 + R_2}{R_1}\right)^2\left|\frac{1 + sC\, R_1 \parallel R_2}{1 + sCR_2}\right|^2 + S_V^{BUF}\left|\frac{sCR_2}{1 + sCR_2}\right|^2,$$

which leads to

$$\overline{V_o^2} \approx S_V^{OA}\frac{\pi}{2}\left(\left(\frac{R_1 + R_2}{R_1}\right)^2 f_p + GBWP - f_z\right) + S_V^{BUF}\frac{\pi}{2}(GBWP - f_p) =$$

$$16 \times 10^{-18}\frac{\pi}{2}\left(25 \times 10^6 + 10^7\right) + 16 \times 10^{-18}\frac{\pi}{2}14 \times 10^6 = 8.8 \times 10^{-10} + 3.5 \times 10^{-10} = (35\ \mu\text{V})^2.$$

### 1.4

We can start from the ideal gain and write

$$V_o(s) = 5\frac{1 + s\tau_z}{1 + s\tau_p}\frac{1}{s} = 5\left(\frac{1}{s} - \frac{\tau_p - \tau_z}{1 + s\tau_p}\right) \Rightarrow v_o(t) = 5\left(1 - \frac{\tau_p - \tau_z}{\tau_p}e^{-t/\tau_p}\right)u(t) = \left(5 - 4e^{-t/\tau_p}\right)u(t),$$

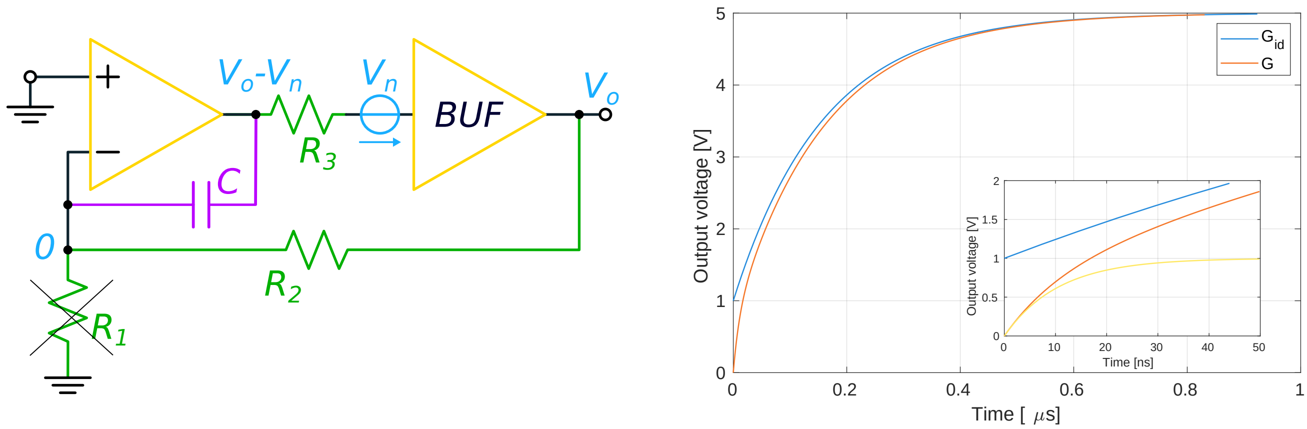



Figure 1: Left = Scheme for buffer noise calculation. Right = Step response according to ideal and real transfers. The inset shows the initial transient with the discussed approximation.

which is plotted in Fig. 1 (right, blue curve). The step-like behavior at $t = 0$ would require an infinite SR, which means that we must consider the real gain, with the extra pole at $f_{0dB}$. Before starting a full analytical treatment, let's try an approximate analysis: we need to evaluate the effect of an extra pole at high frequency (time constant $\tau_0 = 1/(2\pi f_{0dB}) \approx 10.6$ ns) on the previous expression, controlled by $\tau_p = CR_2 = 156$ ns. So, for times comparable to $\tau_0$, the ideal output does not change very much from 1 V, its value at $t = 0^+$. In this range, we can then approximate the true output with the step response:

$$v_o(t) \approx \left(1 - e^{-t/\tau_0}\right) u(t) \Rightarrow \left.\frac{dv_o}{dt}\right|_{max} = \frac{1}{\tau_0} \approx 94 \text{ V}/\mu\text{s}.$$

For the full calculation, it is easier to start from the previous expression, obtaining:

$$V_o(s) = \left(\frac{5}{s} - 4\frac{\tau_p}{1+s\tau_p}\right)\frac{1}{1+s\tau_0} = \frac{5}{s}\frac{1}{1+s\tau_0} - 4\frac{\tau_p}{\tau_p-\tau_0}\left(\frac{\tau_p}{1+s\tau_p} - \frac{\tau_0}{1+s\tau_0}\right).$$

The first term is the LPF step response, and the second is made up of two exponentials, leading to

$$v_o(t) = 5\left(1 - e^{-t/\tau_0}\right)u(t) - 4\frac{\tau_p}{\tau_p-\tau_0}\left(e^{-t/\tau_p} - e^{-t/\tau_0}\right)u(t),$$

from which

$$\left.\frac{dv_o}{dt}\right|_{max} = \left.\frac{dv_o}{dt}\right|_{t=0} = \frac{5}{\tau_0} + 4\frac{\tau_p}{\tau_p-\tau_0}\left(\frac{1}{\tau_p} - \frac{1}{\tau_0}\right) = \frac{1}{\tau_0}.$$

The full output is also shown in Fig. 1 (right), where the inset shows the behavior for short times with the discussed approximation (yellow).

## Problem 2

### 2.1

Not knowing the pulse width, the integration time $T_G$ must account for the worst case, i.e., $T_G = T_{max} = 1\ \mu$s. Setting the output $S/N$ to one, we obtain

$$\frac{S}{N} = \frac{AT}{\sqrt{S_V T_{max}/2}} = 1 \Rightarrow T_{min} = \frac{\sqrt{S_V T_{max}/2}}{A} \approx 14 \text{ ns}.$$

### 2.2

To exploit the full range of the ADC, its maximum input signal (corresponding to $T_{max}$) must be equal to 5 V. The GI gain must therefore be:

$$V_{max} = GAT_{max} = 5 \text{ V} \Rightarrow G = 5 \times 10^9 \text{ s}^{-1}, \Rightarrow V_{min} = GAT_{min} = 70.5 \text{ mV},$$

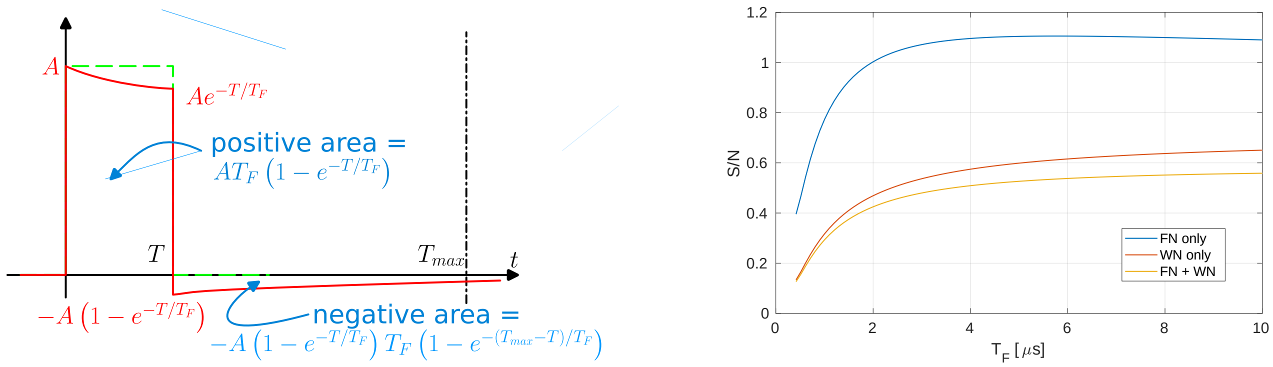



Figure 2: Left = Pulse response of an HPF and integral between 0 and $T_{max}$ of the output signal. Right = $S/N$ as a function of the HPF time constant $T_F$ when different noise contributions are considered.

which defines the LSB. We have then:

$$\frac{5\ \mathrm{V}}{2^n} < V_{min} \Rightarrow n > 6.15,$$

meaning that in practice we need an 8-bit ADC. Please note that setting the condition that the quantization error (that you may have seen in other classes) is smaller than 70.5 mV results in $n > 6.38$.

## 2.3

The output FN contribution is evaluated between $f_{max} = 1/(2T_G)$ and $f_{HP} = 1/(2\pi T_F)$, leading to

$$\overline{n_{FN}^2} = G^2 T_G^2 K \log\left(\frac{\pi T_F}{T_G}\right),$$

while for the output signal we need to consider the HPF pulse response and its integral over a time $T_G = T_{max}$. The signal and its integral are reported in Fig. 2 (left), from which we obtain the GI output signal as

$$V_o = GAT_F\left(e^{T/T_F} - 1\right)e^{-T_{max}/T_F},$$

leading to the expression for $S/N$:

$$\frac{S}{N} = \frac{AT_F\left(e^{T/T_F}-1\right)e^{-T_{max}/T_F}}{T_{max}\sqrt{K\log\left(\frac{\pi T_F}{T_{max}}\right) + S_V\left(\frac{1}{2T_{max}} - \frac{1}{2\pi T_F}\right)}}.$$

## 2.4

The worst case is obviously $T = T_{min} \ll T_F$, so we can expand in series the first exponential term:

$$\left(\frac{S}{N}\right)^2 \approx \frac{A^2}{K}\frac{T_{min}^2}{T_{max}^2}\frac{e^{-2x}}{\log(\pi/x)} = -\frac{A^2}{K}\frac{T_{min}^2}{T_{max}^2}\frac{e^{-2x}}{\log(x/\pi)},$$

where $x = T_{max}/T_F$. Differentiating and zeroing the $x$-dependent part, we get

$$-2e^{-2x}\log\left(\frac{x}{\pi}\right) - \frac{e^{-2x}}{x} = 0 \Rightarrow \log\left(\frac{x}{\pi}\right) = -\frac{1}{2x} \Rightarrow x \approx 0.17 \Rightarrow T_F = 5.8\ \mu\mathrm{s}.$$

The dependence of $S/N$ on $T_F$ is shown in Fig. 2 (right) when different noise contributions are considered.

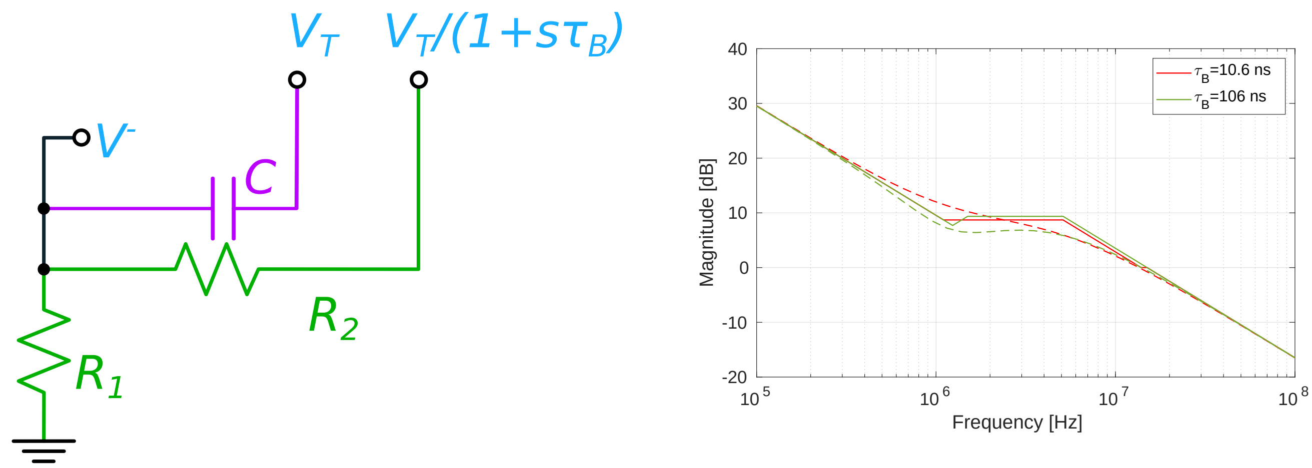



Figure 3: Left = Scheme for loop calculation. Right = $G_{loop}$ for the cases of $\tau_B = 10.6$ ns (15 MHz bandwidth) and 106 ns (1.5 MHz bandwidth).

## Appendix

We can account for the buffer pole by writing its transfer as

$$T = \frac{1}{1+s\tau_B},$$

obtaining the scheme in Fig. 3 (left) for the loop calculation. By linear superposition, we get:

$$\frac{V^-}{V_T} = \frac{R_1 \parallel R_2}{R_1 \parallel R_2 + 1/sC} + \frac{1}{1+s\tau_B}\frac{R_1 \parallel 1/sC}{R_1 \parallel 1/sC + R_2} =$$

$$= \left(\frac{sC\, R_1 \parallel R_2}{1+sC\, R_1 \parallel R_2} + \frac{1}{1+s\tau_B}\frac{R_1}{R_1+R_2+sCR_1R_2}\right) = \frac{R_1}{R_1+R_2}\frac{1}{1+sC\, R_1 \parallel R_2}\left(sCR_2 + \frac{1}{1+s\tau_B}\right)$$

$$= \frac{R_1}{R_1+R_2}\frac{1+sCR_2+s^2CR_2\tau_B}{(1+s\tau_B)(1+sC\, R_1 \parallel R_2)} \Rightarrow G_{loop} = -A(s)\frac{R_1}{R_1+R_2}\frac{1+sCR_2+s^2CR_2\tau_B}{(1+s\tau_B)(1+sC\, R_1 \parallel R_2)}.$$

We can see that $\tau_B$ adds also a zero, which improves stability. We can now consider two limiting cases: For a large-bandwidth buffer, $\tau_B$ is small and we can approximate the numerator as discussed in the class, obtaining:

$$G_{loop} \approx -A(s)\frac{R_1}{R_1+R_2}\frac{(1+sCR_2)(1+s\tau_B)}{(1+s\tau_B)(1+sC\, R_1 \parallel R_2)},$$

*i.e.*, a pole-zero cancellation. This leaves a zero at 1 MHz and a pole at 5 MHz, with phase margin of about 90°.

If instead $\tau_B$ is large (which does not make much sense in reality, as the buffer has to drive the output), the zeros become complex at a frequency $f_z = 1/(2\pi\sqrt{CR_2\tau_B})$, that could fall after the $\tau_B$ pole and eventually not being able to increase the phase margin. However, this happens only for $\tau_B \gg CR_2$ (for $\tau_B = 10CR_2$ the phase margin is still 84°).