Electronics – 96032

POLITECNICO DI MILANO

Laplace Transforms, Linear Circuits and Bode Plots

Alessandro Spinelli
Phone: (02 2399) 4001
alessandro.spinelli@polimi.it spinelli.faculty.polimi.it

Disclaimer

Slides are supplementary material and are NOT a replacement for textbooks and/or lecture notes

Outline

- Laplace transforms
- Application to LTI circuits
- Bode plots
- Appendix: Complex poles

Laplace transform

$$F(s) = \int_0^{+\infty} f(t) e^{-st} dt = \mathcal{L}(f)$$

$\mathcal{L}$ is a linear operator:

$$\mathcal{L}(\alpha f(t) + \beta g(t)) \; = \alpha\mathcal{L}(f) + \beta\mathcal{L}(g)$$

Note: in the following, all signals are defined only for $t \geq 0$, i.e., they are (implicitely) multiplied by the unit step function $u(t)$

Properties

1. Time differentiation:

$$\mathcal{L}\big(f'(t)\big) = sF(s) - f(0)$$

- If we think in terms of distributional derivatives (e.g., Dirac delta as derivative of step function), then $f(0) \rightarrow f(0^-)$ $(= 0$ in our case)
- If we think in terms of classical derivatives (e.g., zero as derivative of step function), then $f(0) \rightarrow f(0^+)$

Properties

2. “Frequency” differentiation:

$$\mathcal{L}\big(tf(t)\big) = -\frac{dF(s)}{ds}$$

- More generally

$$\mathcal{L}\big(t^n f(t)\big) = (-1)^n \frac{d^n F(s)}{ds^n}$$

Properties

3. Time integration:

$$\mathcal{L}\left(\int_0^t f(\tau)d\tau\right) = \frac{F(s)}{s}$$

4. Frequency integration:

$$\mathcal{L}\left(\frac{f(t)}{t}\right) = \int_s^\infty F(\sigma)d\sigma$$

Properties

5. Time shifting:

$$\mathcal{L}\big(f(t-T)\big) = e^{-sT}F(s)$$

6. Frequency shifting:

$$\mathcal{L}\big(e^{at}f(t)\big) = F(s-a)$$

7. Convolution:

$$\mathcal{L}(f(t) * g(t)) = F(s)G(s)$$

   - Dual properties is more complicated, involving integral in complex domain

Properties

8. Initial value theorem:

$$f(0^+) = \lim_{s\to\infty} sF(s)$$

9. Final value theorem:

$$f(+\infty) = \lim_{s\to 0} sF(s)$$

Both can be extended to compute the values of the derivatives of f just by recalling #1. For first derivatives, this means

$$f'(0^+) = \lim_{s\to\infty} s^2F(s) \qquad f'(+\infty) = \lim_{s\to 0} s^2F(s)$$

Properties

10.Scaling

$$\mathcal{L}(f(at)) = \frac{1}{|a|}F\left(\frac{s}{a}\right)$$

In particular, for $a = -1$ (time reversal) we get:

$$\mathcal{L}(f(-t)) = F(-s)$$

Elementary signals – 1

- Dirac delta function:

$$\mathcal{L}\big(\delta(t)\big) = \int_{0}^{+\infty} \delta(t)e^{-st}dt = 1$$

- Heaviside step function (integral of $\delta(t)$):

$$\mathcal{L}\big(u(t)\big) = \frac{1}{s}$$

- Ramp function (integral of $u(t)$)

$$\mathcal{L}(t) = \frac{1}{s^2}$$

Also obtained from #2

Elementary signals – 2

- Rectangle function:

$$\mathcal{L}\big(\mathrm{rect}(0,T)\big)=\mathcal{L}\big(u(t)-u(t-T)\big)=\frac{1-e^{-sT}}{s}$$

- Decreasing exponential function (from #6, with $f(t)=u(t)$ and $a=-1/\tau$):

$$\mathcal{L}\big(e^{-t/\tau}u(t)\big)=\frac{1}{s+1/\tau}=\frac{\tau}{1+s\tau}$$

Differentiation example

$$f(t) = e^{-t/\tau} \Rightarrow F(s) = \frac{\tau}{1+s\tau} \qquad \mathcal{L}\big(f'(t)\big) = sF(s) = \frac{s\tau}{1+s\tau}$$

In fact, in a distribution frame:

$$f'(t) = \delta(t) - \frac{1}{\tau}e^{-t/\tau}$$

$$\mathcal{L}\big(f'(t)\big) = 1 - \frac{1}{\tau}\frac{\tau}{1+s\tau} = \frac{s\tau}{1+s\tau}$$

Elementary signals – 3

- Sine function

$$\mathcal{L}(\sin(\omega t)) = \frac{\omega}{s^2 + \omega^2}$$

- Cosine function (from #1)

$$\mathcal{L}(\cos(\omega t)) = \frac{s}{s^2 + \omega^2}$$

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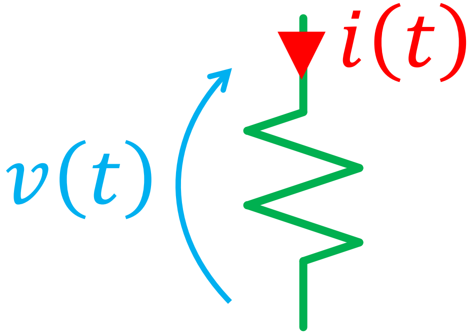
Application to LTI systems

$$\begin{tikzcd}
x(t) \arrow[r] \arrow[d, "\mathcal{L}"'] & \boxed{h(t)} \arrow[r] \arrow[d, "\mathcal{L}"'] & y(t) = x(t) * h(t) \\
X(s) \arrow[r] & \boxed{H(s)} \arrow[r] & Y(s) = X(s)H(s) \arrow[u, "\mathcal{L}^{-1}"]
\end{tikzcd}$$

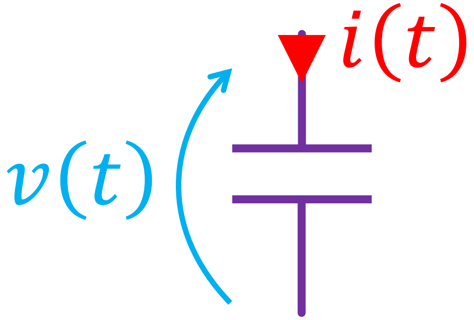
R, L, C



$$v(t) = Ri(t)$$

$$V(s) = RI(s)$$

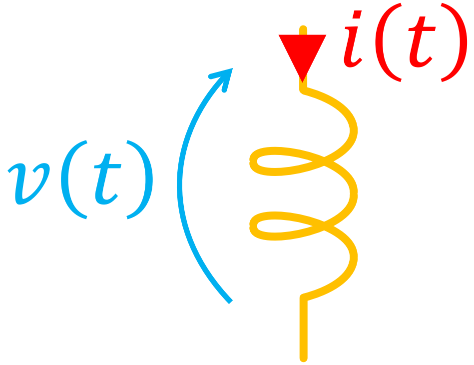
$$\frac{V(s)}{I(s)} = R$$



$$i(t) = C\frac{dv(t)}{dt}$$

$$I(s) = sCV(s)$$

$$\frac{V(s)}{I(s)} = \frac{1}{sC} = Z_C$$

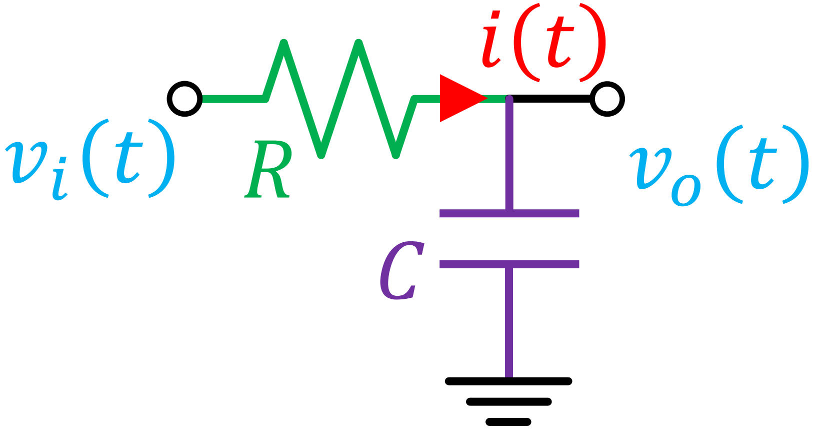


$$v(t) = L\frac{di(t)}{dt}$$

$$V(s) = sLI(s)$$

$$\frac{V(s)}{I(s)} = sL = Z_L$$

RC network – time domain



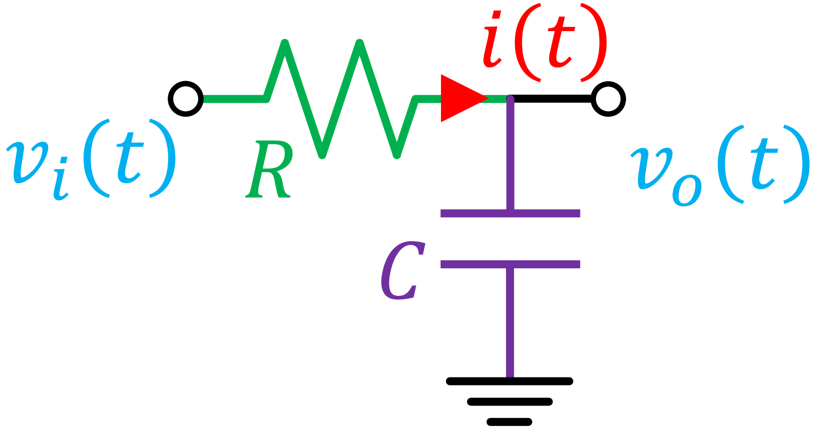
$$\begin{cases} v_i = v_o + Ri \\ i = C\dfrac{dv_o}{dt} \end{cases} \Rightarrow RC\frac{dv_o}{dt} + v_o = v_i$$

$$RCz + 1 = 0 \Rightarrow z = -\frac{1}{RC} \Rightarrow v_0 = Ke^{-\frac{t}{RC}}$$ (homogeneous sol.)

$$v_i = Au(t) \Rightarrow v_o = A$$ (particular solution)

$$v_o(0) = 0 \Rightarrow K = -A \Rightarrow v_o = A\left(1 - e^{-\frac{t}{RC}}\right)$$

RC network – Laplace domain



Single, real pole
in $s = -1/CR$

$$\frac{V_o(s)}{V_i(s)} = \frac{1/sC}{R + 1/sC} = \frac{1}{1 + sCR}$$

Poles are the eigenvalues of the characteristic equation $\Rightarrow$ they represent the time constants of the output

Output signals

- Delta function response, $v_i(t) = A\delta(t)$

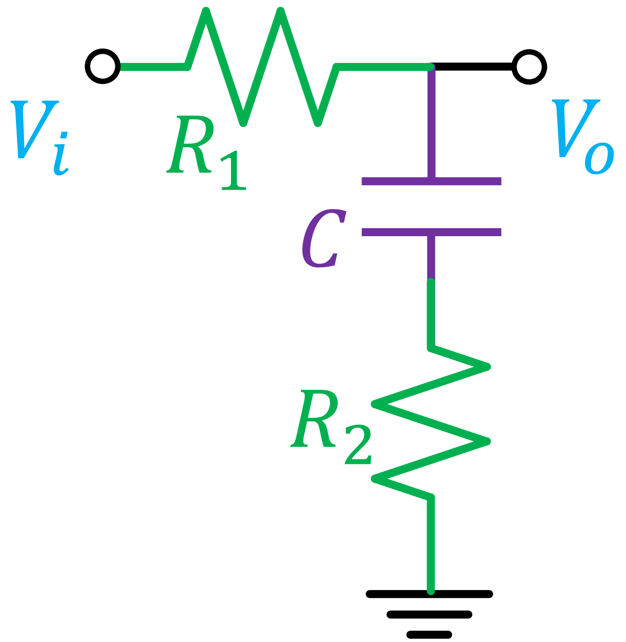
$$V_i(s) = A \Rightarrow V_o = \frac{A}{1 + sCR} = \frac{A}{\tau}\frac{\tau}{1 + s\tau} \Rightarrow v_o = \frac{A}{\tau}e^{-t/\tau}$$

- Step function response, $v_i(t) = Au(t)$

$$V_i(s) = \frac{A}{s} \Rightarrow V_o = \frac{A}{1 + s\tau}\frac{1}{s} = A\left(\frac{1}{s} - \frac{\tau}{1 + s\tau}\right) \Rightarrow$$

$$v_o = A\left(u(t) - e^{-t/\tau}u(t)\right) = A(1 - e^{-t/\tau})u(t)$$

Ex: Lag network (step response)



$$\frac{V_o}{V_i} = \frac{R_2 + 1/sC}{R_1 + R_2 + 1/sC} = \frac{1 + sCR_2}{1 + sC(R_1 + R_2)}$$

$$V_o = \frac{1 + sCR_2}{1 + sC(R_1 + R_2)} \frac{A}{s}$$

$$V_o = A\left(\frac{1}{s} - \frac{CR_1}{1 + sC(R_1 + R_2)}\right) \Rightarrow v_o(t) = A\left(1 - \frac{R_1}{R_1 + R_2} e^{-t/\tau}\right)$$

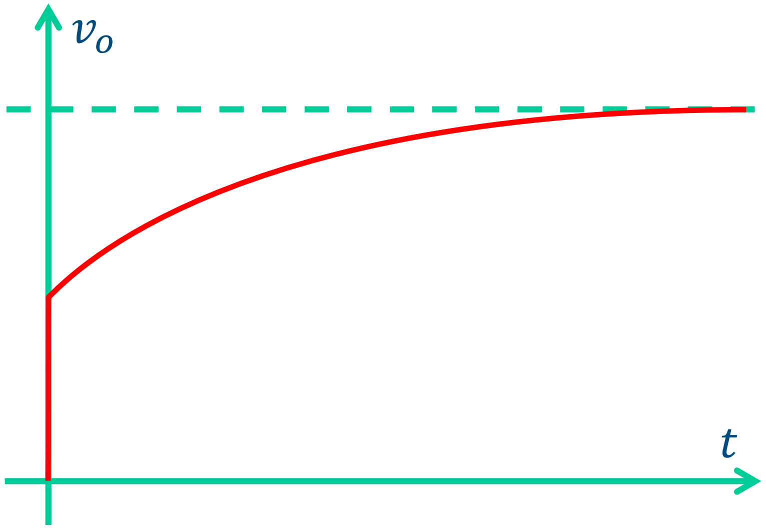
Alternative (rough) method

1) Compute:

$$v_o(0) = \lim_{s \to \infty} sV_o(s) = A\frac{R_2}{R_1 + R_2}$$

$$v_o(\infty) = \lim_{s \to 0} sV_o(s) = A$$

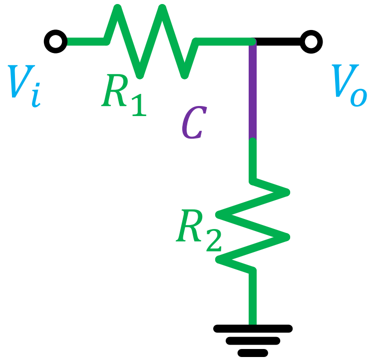
2) Connect extremes with an exponential curve having time constant determined by the single pole



One more approach

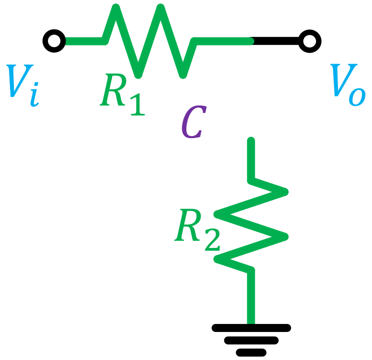
Voltage across a capacitor cannot change abruptly (unless you are applying a δ-function current)

- $v_c(0^+) = v_c(0^-) = 0$
- $v_0(0^+) = \frac{AR_1}{R_1+R_2}$



For $t \to \infty$, the capacitor behaves as an open circuit

- $v_o(\infty) = A$



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LTI systems and sinusoidal inputs

- The output of a stable LTI system to a sinusoidal input contains a transient and a steady-state term
- The steady-state term is also sinusoidal at the same frequency as the input, but with different amplitude and phase:

$$A\sin(\omega t) \longrightarrow \boxed{T(s)} \longrightarrow AB\sin(\omega t + \phi)$$

where

$$B = |T(j\omega)| \qquad \phi = \measuredangle(T(j\omega))$$

Bode plots

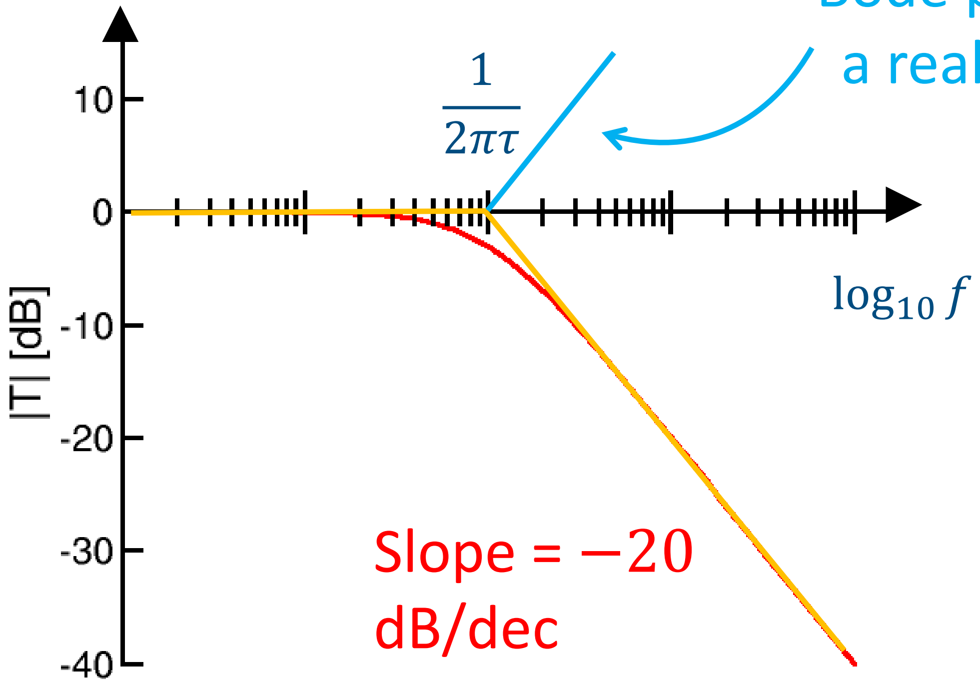
- Provide an efficient way to plot the frequency response (magnitude and phase) of LTI systems
- We consider asymptotic Bode plots, i.e., piecewise linear approximations on a suitable scale:
  - Magnitude: $\mathrm{dB} = 20\log_{10}|\cdot|$
  - Phase: linear scale
  - Frequency: log scale

Single real pole: magnitude

$$T(s) = \frac{1}{1+s\tau} \Rightarrow |T(j\omega)|_{dB} = 20\log_{10}\left(\frac{1}{\sqrt{1+(\omega\tau)^2}}\right)$$

$$= -20\log_{10}\sqrt{1+(\omega\tau)^2}$$

$$\approx \begin{cases} 0 & \omega\tau \ll 1 \\ -20\log_{10}\omega\tau & \omega\tau \gg 1 \end{cases}$$

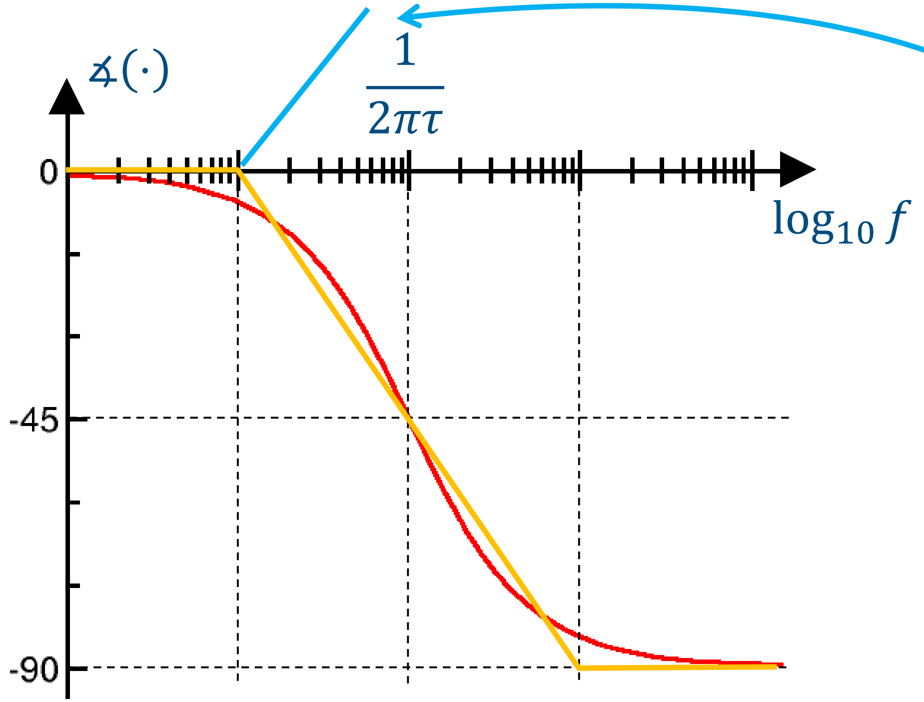


Bode plot of a real zero

Slope = -20 dB/dec

Single real pole: phase

$$\measuredangle(T(j\omega)) = \measuredangle\left(\frac{1}{1+j\omega\tau}\right) = -\arctan(\omega\tau)$$



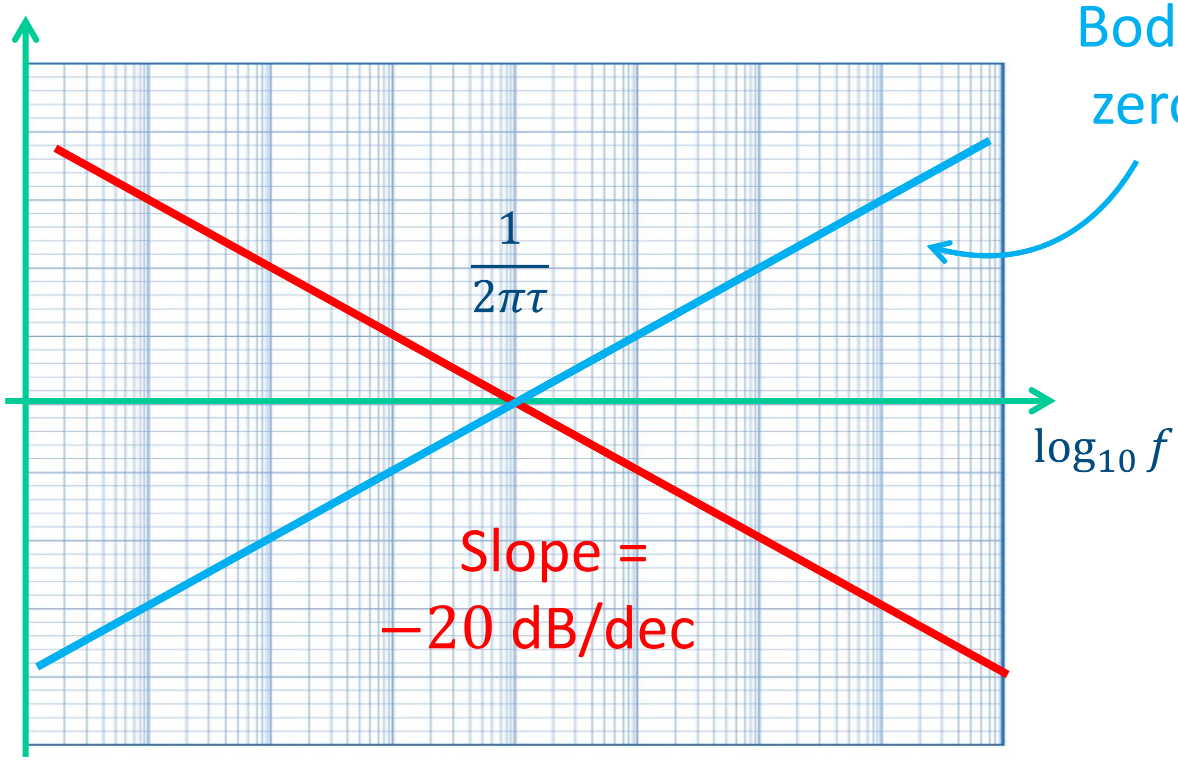
Bode plot of a real zero

Pole (zero) in $s = 0$

$$T(s) = \frac{1}{s\tau} \Rightarrow |T(j\omega)|_{dB} = 20\log_{10}\left(\frac{1}{\omega\tau}\right) = -20\log_{10}\omega\tau$$

x-axis intercept:

$$|T(j\omega)| = 1 \Rightarrow \omega = \frac{1}{\tau}$$



Slope = -20 dB/dec

Bode plot of a zero in $s = 0$

General case (real singularities)

- $T(s)$ can always be expressed as

$$T(s) = G\,\frac{(1+s\tau_{z1})(1+s\tau_{z2})\ldots(1+s\tau_{zn})}{(1+s\tau_{p1})(1+s\tau_{p2})\ldots(1+s\tau_{pm})}$$

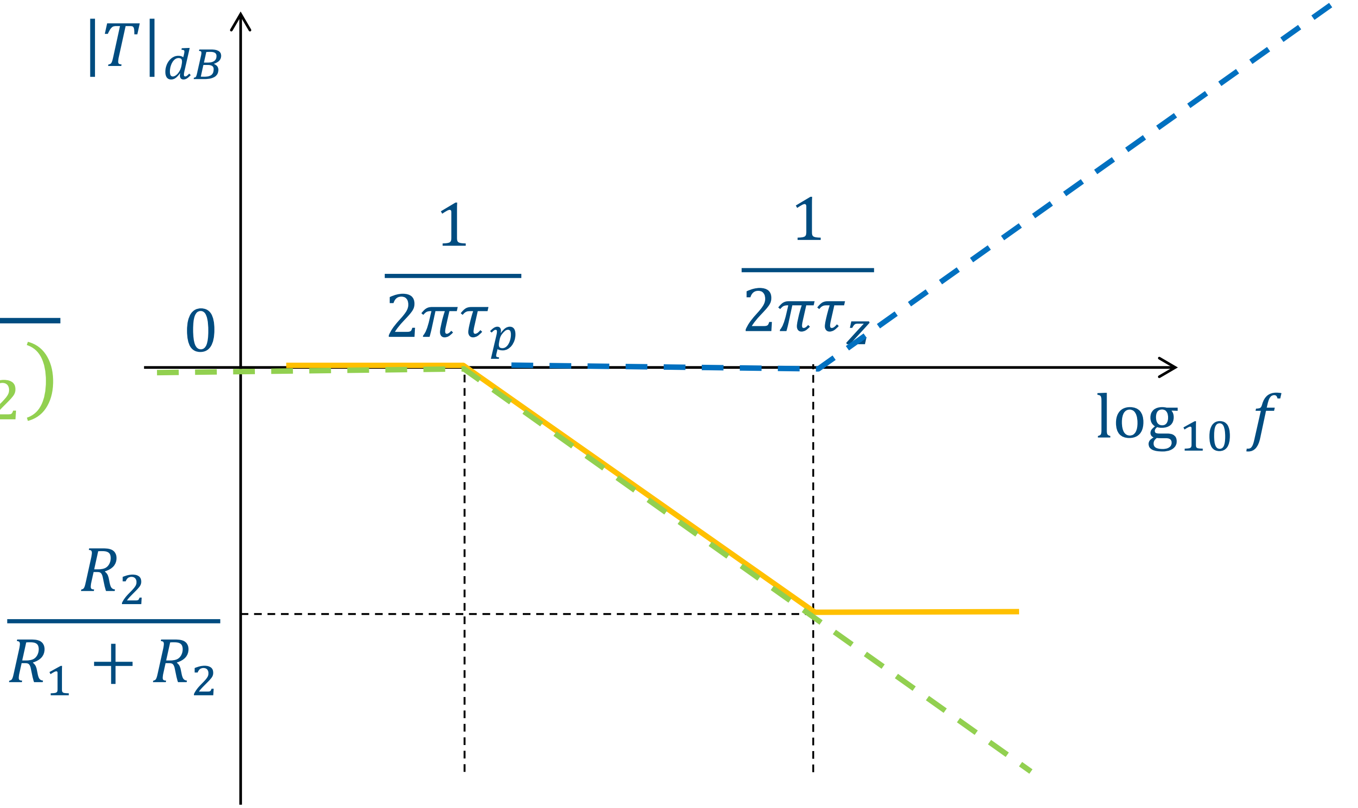
- Thanks to log and arctan properties:

$$|T|_{dB} = G_{dB} + \sum_{i=1}^{n} |1+j\omega\tau_{zi}|_{dB} - \sum_{j=1}^{m} \left|1+j\omega\tau_{pj}\right|_{dB}$$

$$\measuredangle(T) = \sum_{i=1}^{n} \measuredangle(1+j\omega\tau_{zi}) - \sum_{j=1}^{m} \measuredangle(1+j\omega\tau_{pj})$$

Ex: Lag network

$$T(s) = \frac{1 + sC\overbrace{R_2}^{\tau_z}}{1 + s\underbrace{C(R_1 + R_2)}_{\tau_p}}$$

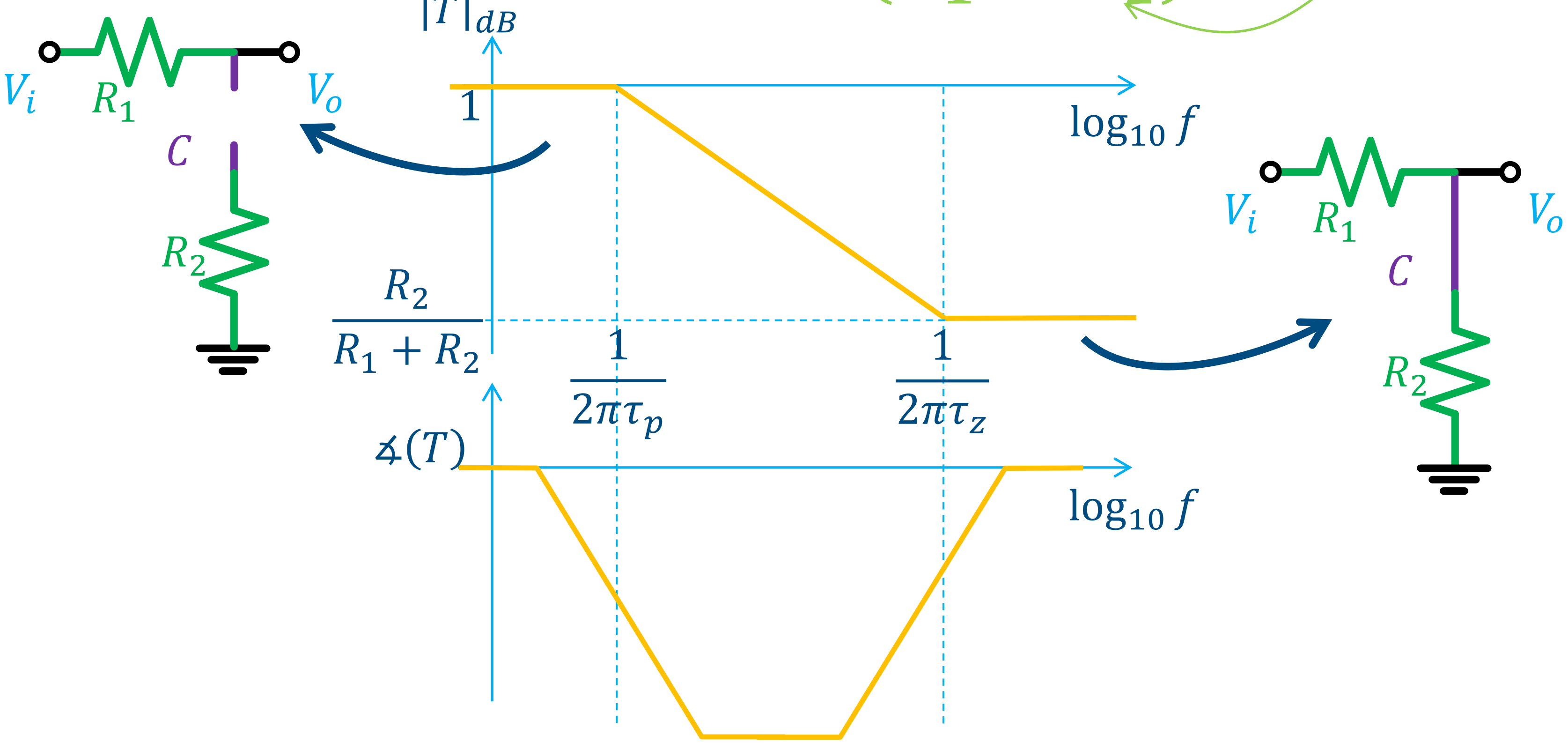


In practice…

- Magnitude:
  - At every zero frequency, the slope is increased by 1 (20 dB/dec)
  - At every pole frequency, the slope is reduced by 1 (20 dB/dec)
- Phase:
  - Composition is a bit more complicated, but a rough view can be obtained by abruptly adding $\pm 90°$ at every zero/pole frequency

Asymptotic values

$$T(s) = \frac{1 + s\overbrace{CR_2}^{\tau_z}}{1 + s\underbrace{C(R_1 + R_2)}_{\tau_p}}$$



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Complex poles

- We consider a second order LPF having

$$T(s) = \frac{\omega_p^2}{s^2 + 2\xi\omega_p s + \omega_p^2} = \frac{\omega_p^2}{s^2 + s\left(\frac{\omega_p}{Q}\right) + \omega_p^2} \Rightarrow Q = \frac{1}{2\xi}$$

where

ξ= damping factor (mostly used in system control)

Q = quality factor (mostly used in electronics/filter design)

Quality factor and poles

- The roots are

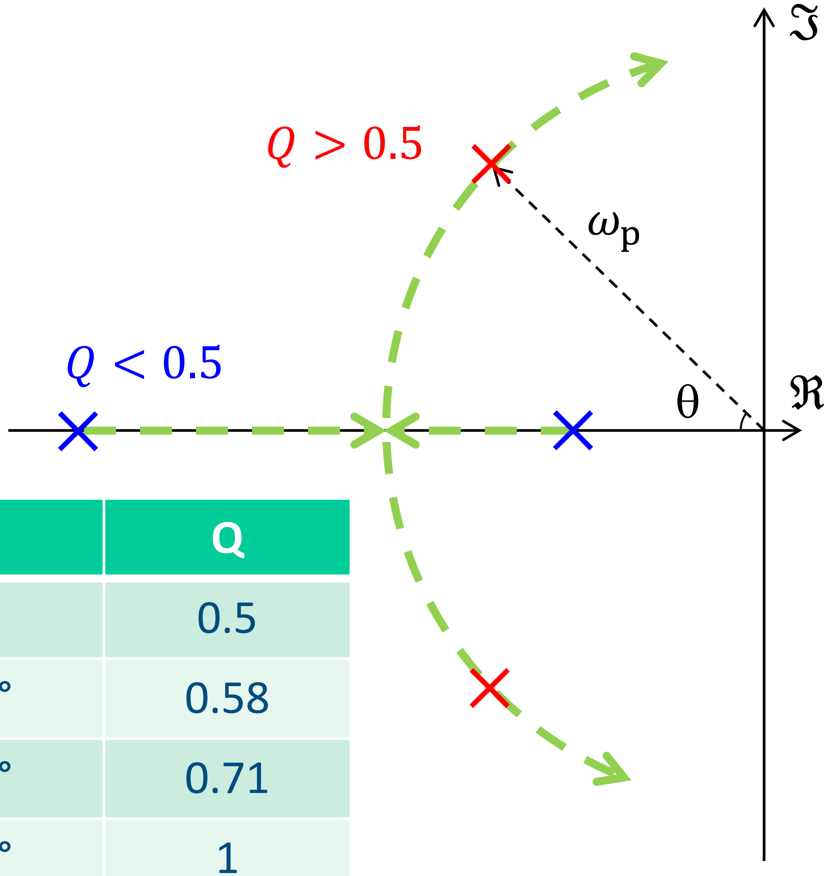
$$\omega = -\frac{\omega_p}{2Q} \pm \omega_p \sqrt{\frac{1}{4Q^2} - 1} = -\frac{\omega_p}{2Q}\left(1 \pm \sqrt{1 - 4Q^2}\right)$$

$Q < 0.5 \Rightarrow$ poles are real and separated

$Q = 0.5 \Rightarrow$ poles are real and coincident

$Q > 0.5 \Rightarrow$ poles are complex conjugated

Complex poles



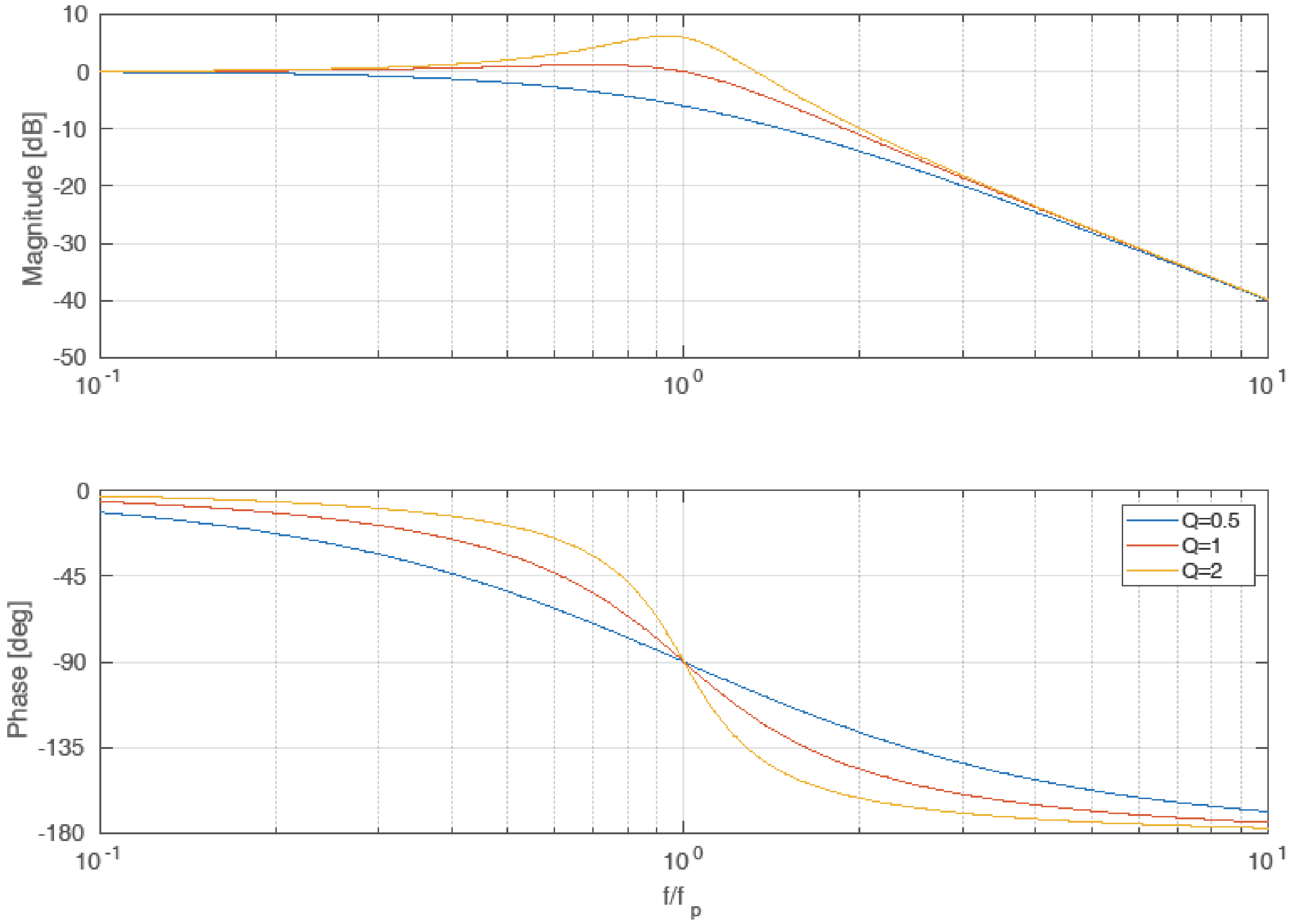
θ	Q
0°	0.5
30°	0.58
45°	0.71
60°	1
75°	1.93

- For $Q > 0.5$ we have
  - A real part equal to $-\frac{\omega_p}{2Q}$
  - The frequency of the poles is $\omega_p/2\pi$
  - At $\omega = \omega_p$ we have

$$T(j\omega_p) = \frac{\omega_p^2}{(j\omega_p)^2 + j\omega_p\left(\frac{\omega_p}{Q}\right) + \omega_p^2} = -jQ$$

$$|T(j\omega_p)| = Q$$

Bode diagrams ($Q \geq 0.5$)



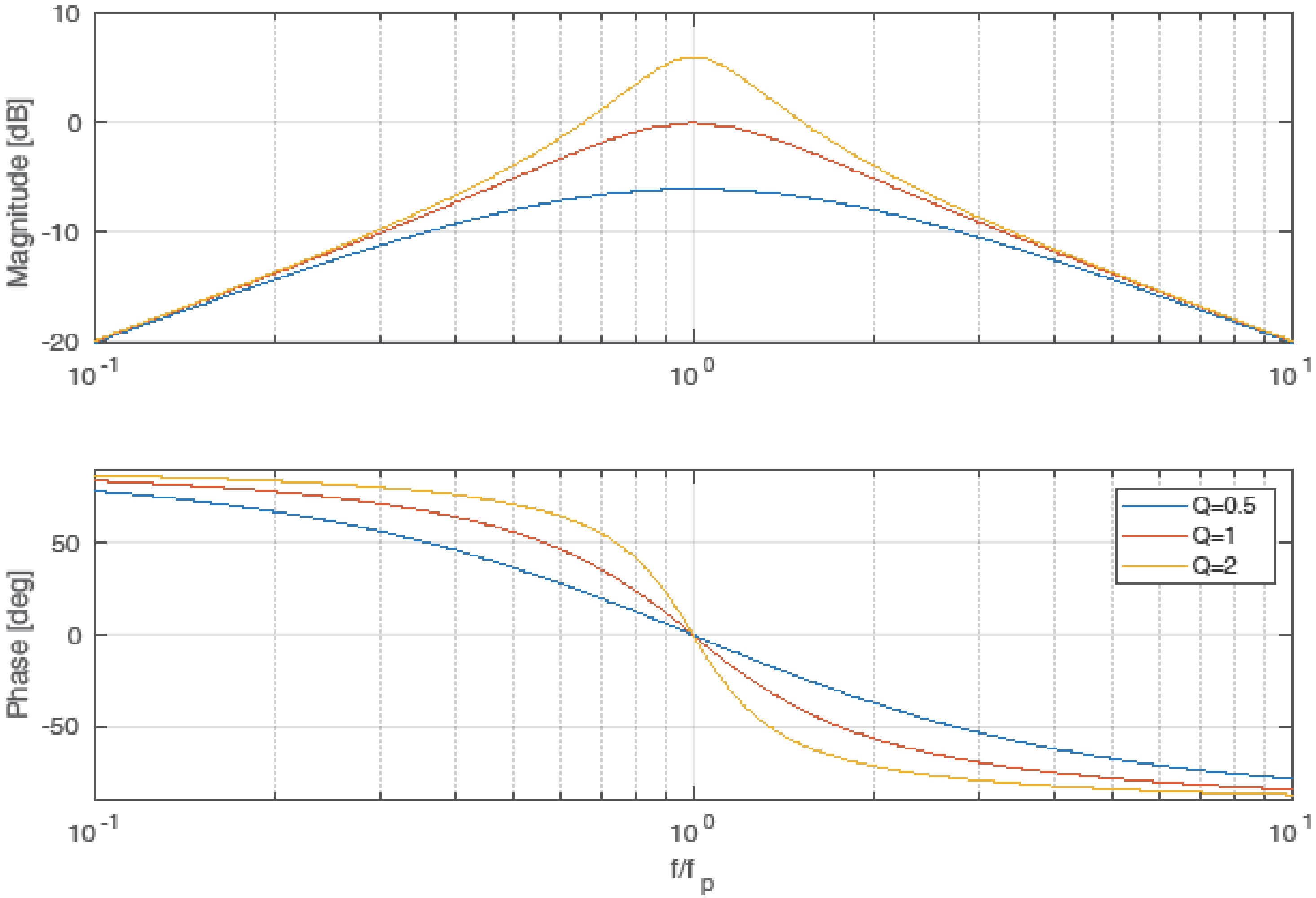
Band-pass filter

- The TF of a second-order BPF is

$$T(s) = \frac{s\omega_p}{s^2 + s\left(\frac{\omega_p}{Q}\right) + \omega_p^2}$$

- Pole position is the same as the LPF’s
- We still have $\left|T\left(j\omega_p\right)\right| = Q$
- The -3dB BW is $\frac{f_p}{Q}$

Bode diagrams ($Q \geq 0.5$)



Homework

1. Compute the step response of a CR filter
2. Design a simple network (made with resistors and capacitors only) having a zero and a pole with $f_z < f_p$ (a lead network)
3. Consider a BP filter with real poles, having

$$T(s) = \frac{As\tau_1}{(1+s\tau_1)(1+s\tau_2)} \qquad (\tau_1 \gg \tau_2)$$

Draw its quantitative Bode diagram and work out the step response