Electronics – 96032

POLITECNICO DI MILANO

Noise Transfer in Circuits

Alessandro Spinelli
Phone: (02 2399) 4001
alessandro.spinelli@polimi.it spinelli.faculty.polimi.it

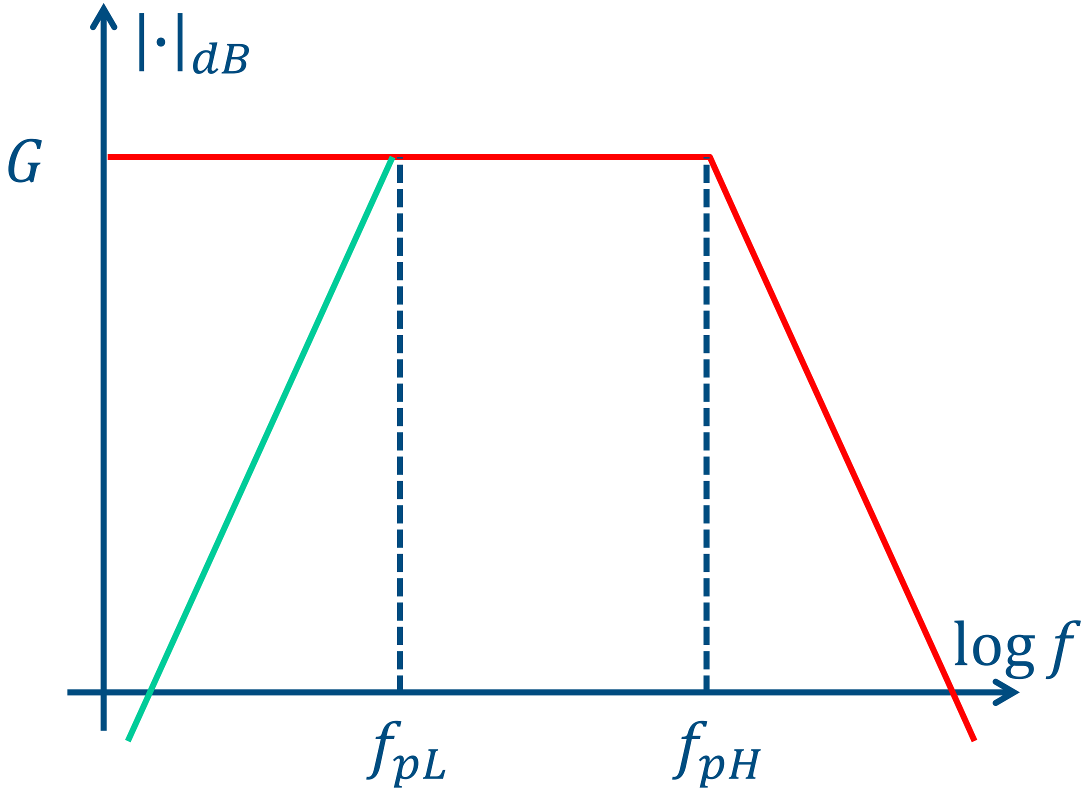
Disclaimer

Slides are supplementary material and are NOT a replacement for textbooks and/or lecture notes

Outline

- Inverting amplifier
- Integrator
- Differentiator
- Appendix: BP filter and white noise

Review



- LP filter noise transfer:

$$\overline{n^2} = G^2 S_{in} \frac{\pi}{2} f_{pH}$$

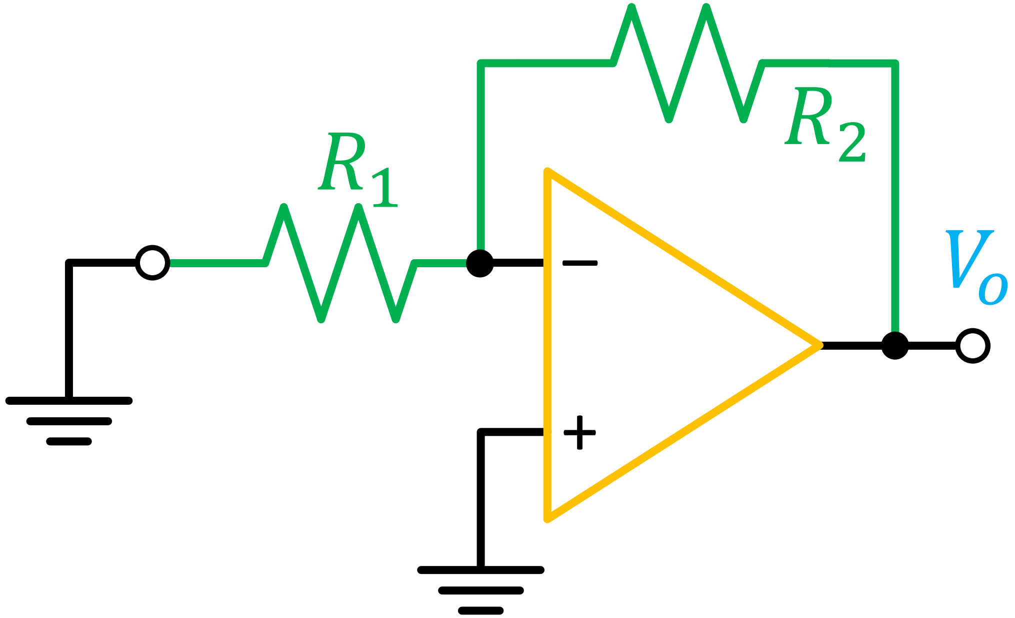
- BP filter noise transfer:

$$\overline{n^2} = G^2 S_{in} \frac{\pi}{2} \left(f_{pH} - f_{pL}\right)$$

$$\approx G^2 S_{in} \frac{\pi}{2} f_{pH}$$

(see Appendix)

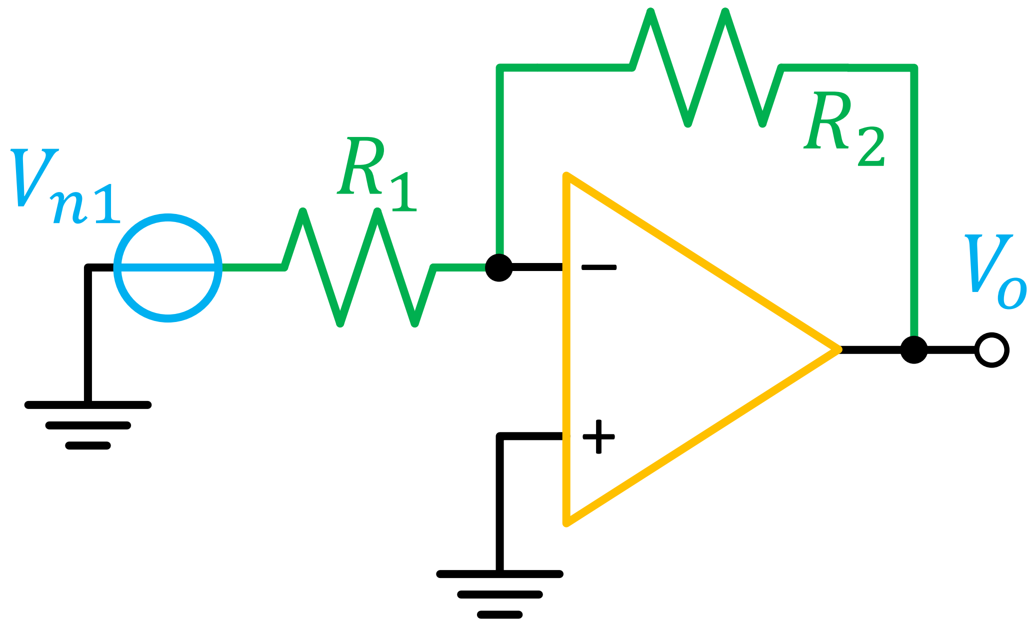
Inverting amplifier



- $R_1 = 1\ \text{k}\Omega,\ \ R_2 = 100\ \text{k}\Omega$
- $GBWP = 10\ \text{MHz}$
- $\sqrt{S_V} = 10\ \text{nV}/\sqrt{\text{Hz}}$
- $\sqrt{S_I} = 10\ \text{pA}/\sqrt{\text{Hz}}$

1. Find the total output noise

R_1 noise



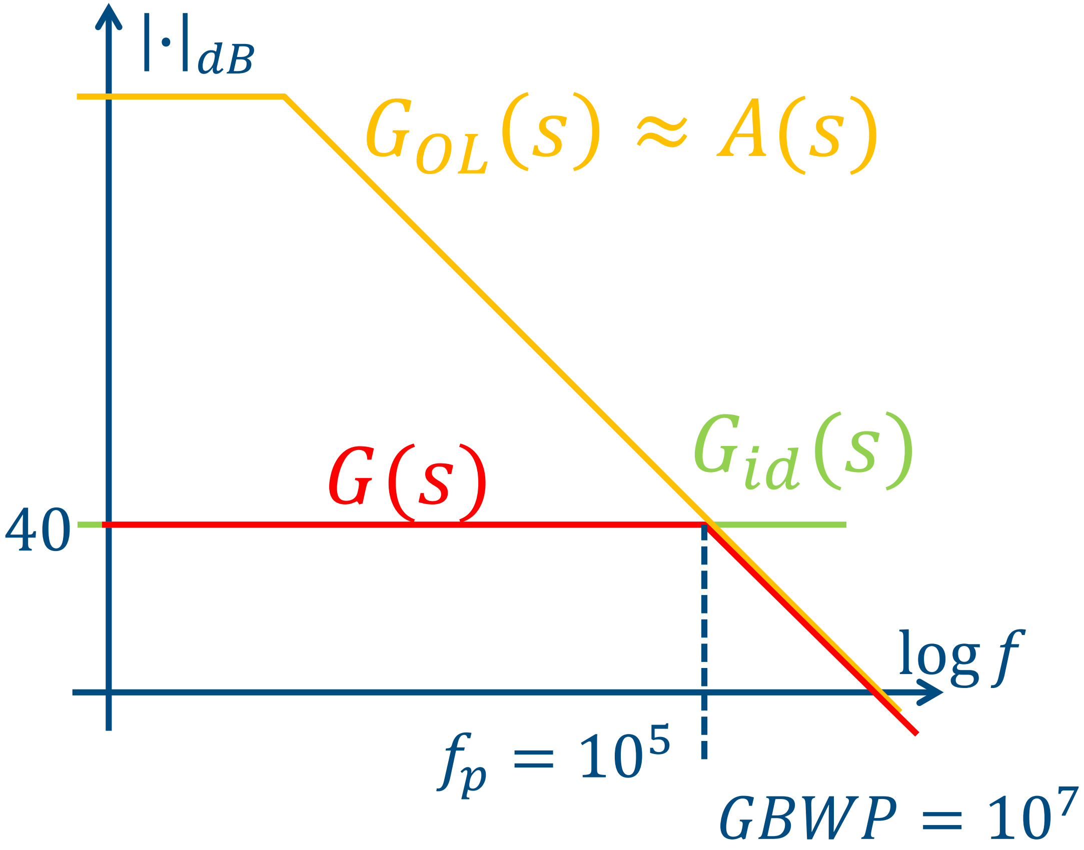
$$V_o = -\frac{R_2}{R_1} V_{n1}$$

$$S_{V_o} = \left(\frac{R_2}{R_1}\right)^2 S_V$$

$$\overline{V_o^2} = \int S_{V_o} df$$

Total output noise diverges $\Rightarrow$ we need to consider the true high-frequency transfer!

Gain



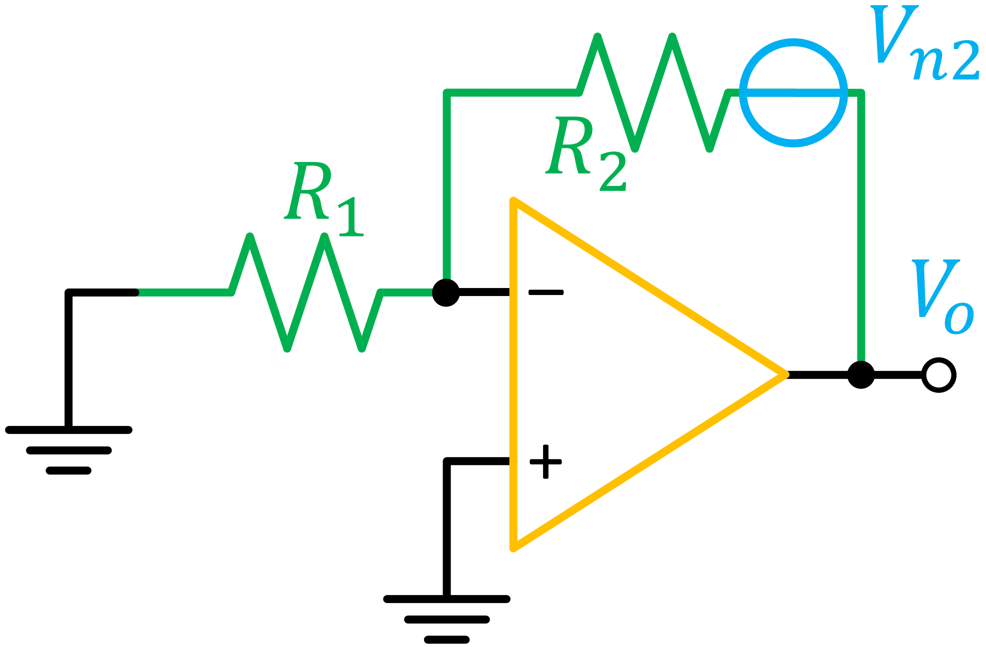
$$G_{OL} = -A(s)\frac{R_2}{R_1 + R_2} \approx -A(s)$$

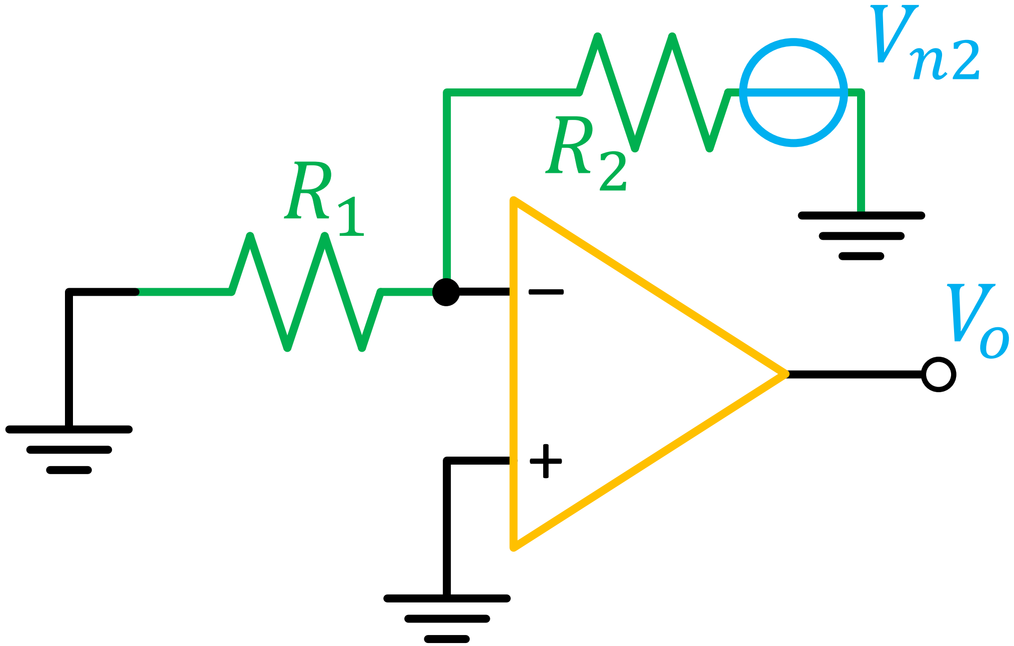
$$|G| = \frac{R_2/R_1}{1 + s\tau_p} \qquad \frac{1}{2\pi\tau_p} = 10^5 \text{ Hz}$$

$$\overline{V_o^2} = 4k_B T R_1 \left(\frac{R_2}{R_1}\right)^2 \frac{\pi}{2} f_p$$

$$\approx 2.59 \times 10^{-8}\ \text{V}^2$$

R_2 noise

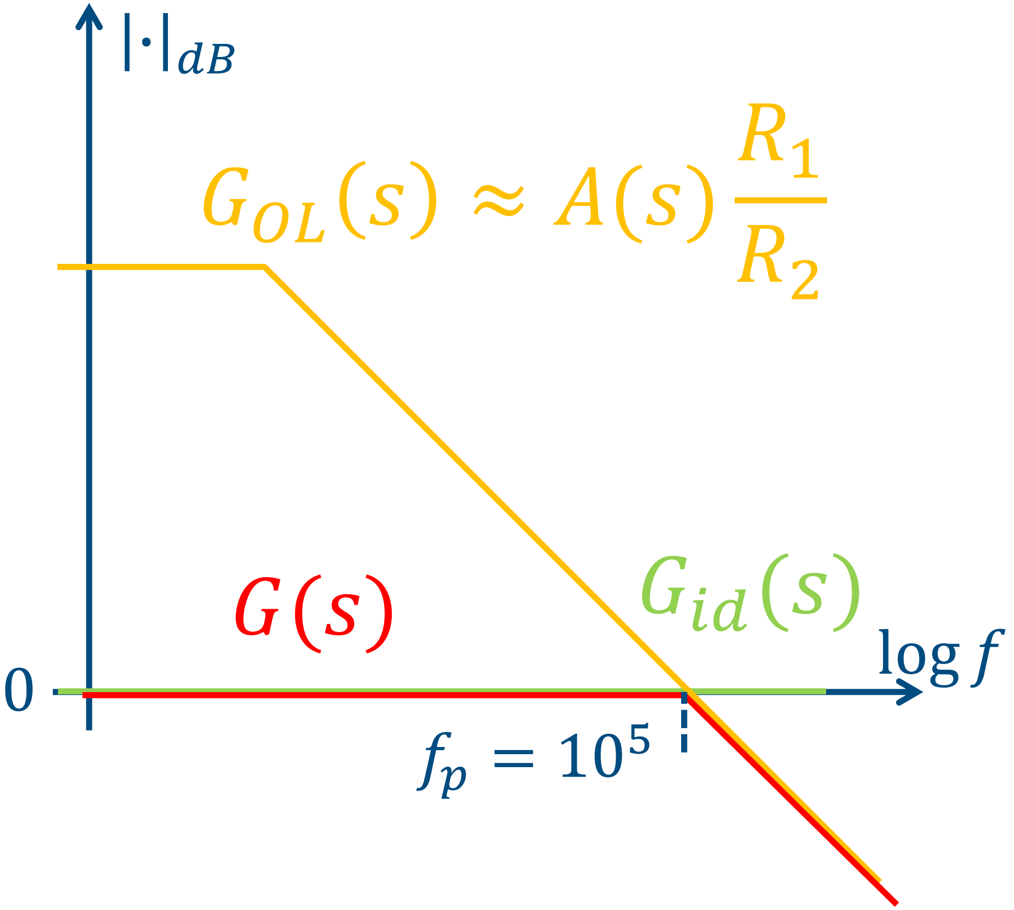




$$V_o = V_{n2} \Rightarrow S_{V_o} = S_V$$

$$G_{OL} = -A(s)\frac{R_1}{R_1 + R_2}$$

Gain



$$G_{OL} \approx -A(s)\frac{R_1}{R_2} \qquad |G| = \frac{1}{1+s\tau_p}$$

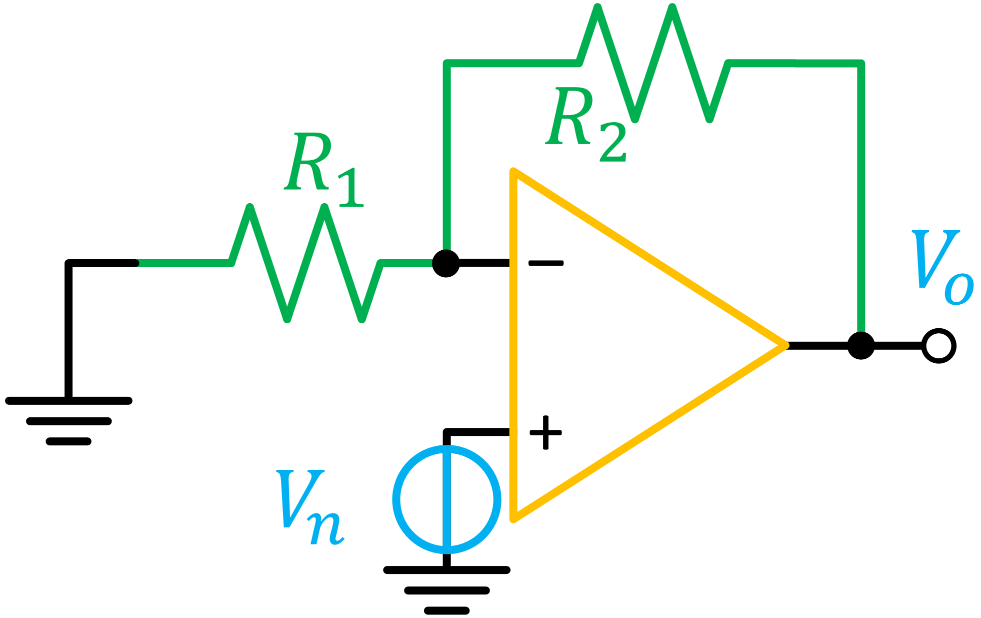
$$\frac{1}{2\pi\tau_p} = GBWP\frac{R_1}{R_2} = 10^5\ \text{Hz}$$

$$\overline{V_o^2} = 4k_BTR_2\frac{\pi}{2}f_p$$

$$\approx 2.59 \times 10^{-10}\ \text{V}^2$$

(smaller than R_1 noise)

OpAmp voltage noise

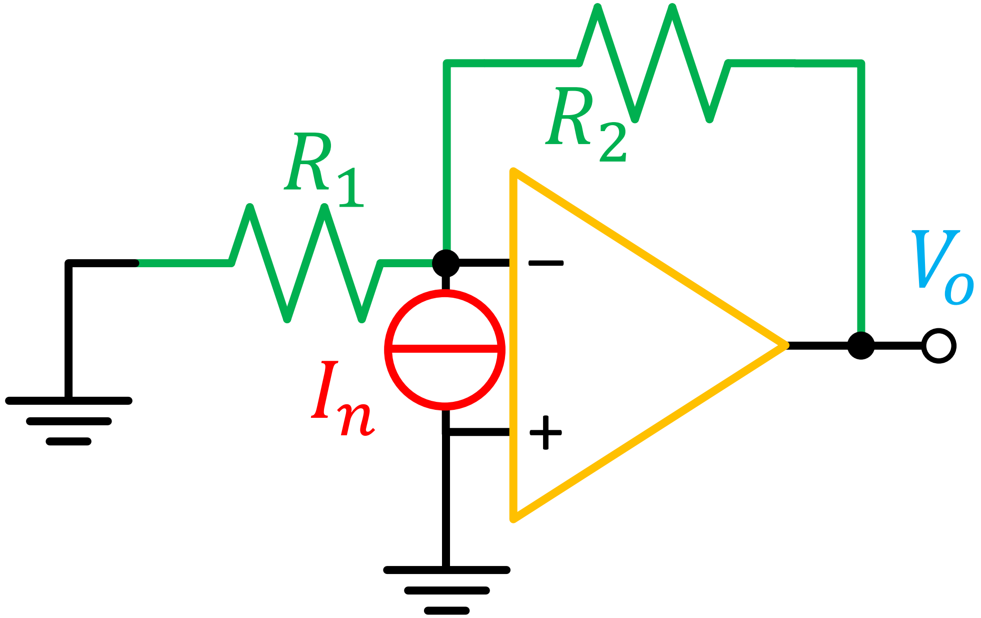


$$V_o = V_n \frac{R_1 + R_2}{R_1}$$

The pole added by the OA is always at $f_p = 100$ kHz (0dB frequency of G_{loop})

$$\overline{V_o^2} = S_V \left(\frac{R_1 + R_2}{R_1}\right)^2 \frac{\pi}{2} f_p$$
$$\approx 1.59 \times 10^{-7}\ \mathrm{V}^2$$

OpAmp current noise



$$V_o = I_n R_2$$

$$\overline{V_o^2} = S_I R_2^2 \frac{\pi}{2} f_p$$

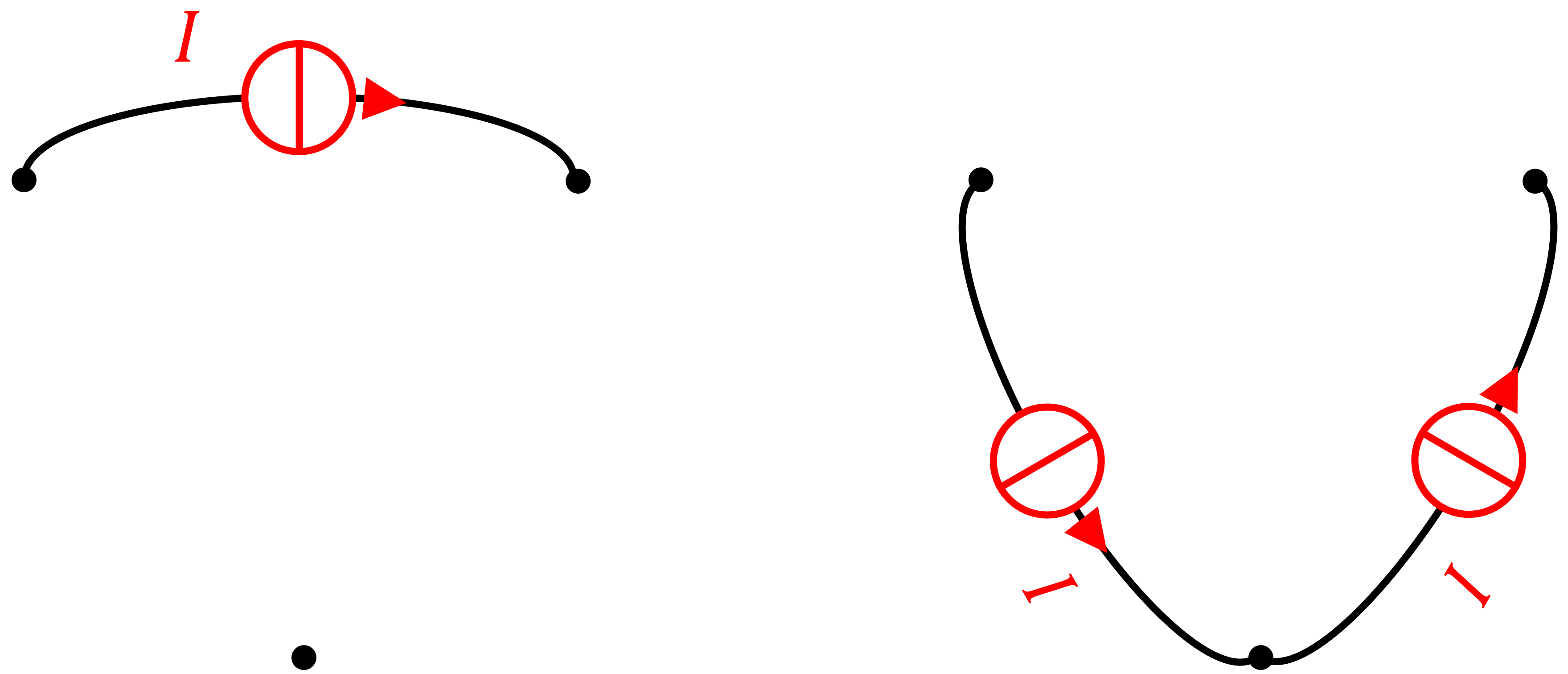
$$\approx 1.57 \times 10^{-7}\ \text{V}^2$$

The total output noise is

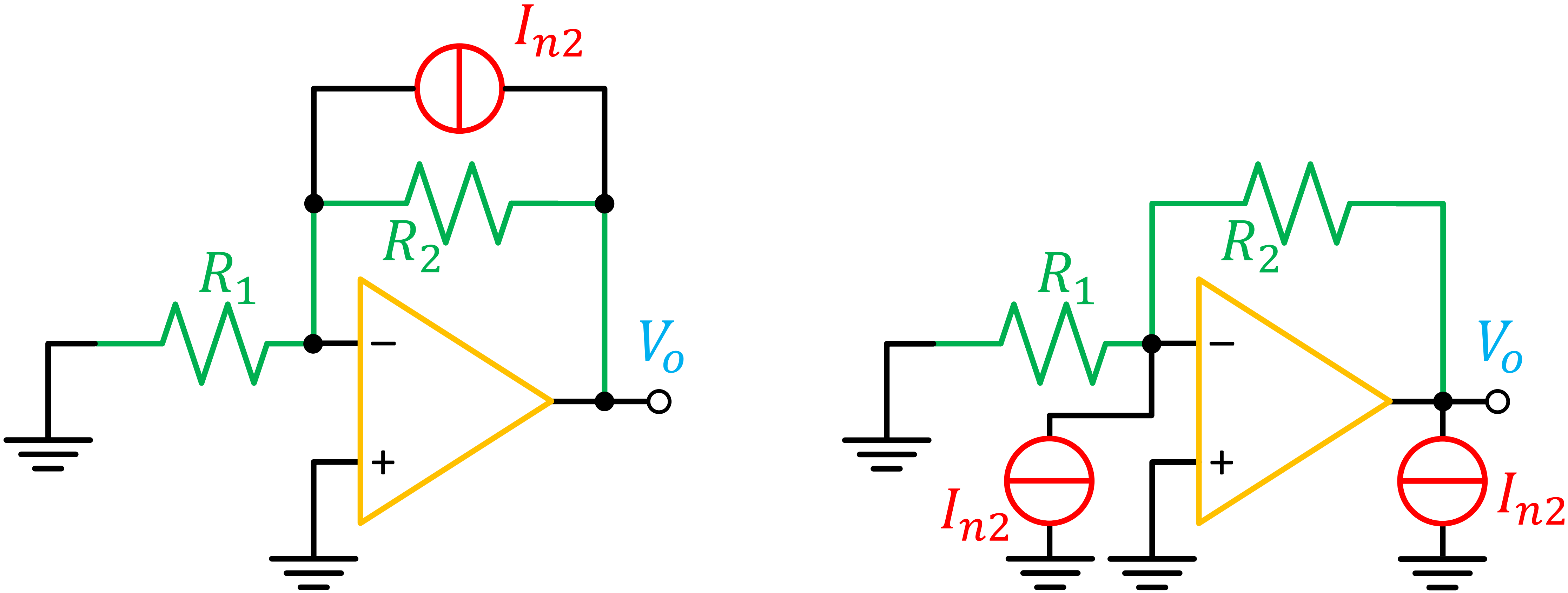
$$\overline{V_o^2} = 2.59 \times 10^{-8} + 2.59 \times 10^{-10} + 1.59 \times 10^{-7} + 1.57 \times 10^{-7}$$

$$\approx 3.42 \times 10^{-7}\ \text{V}^2 \Rightarrow \sqrt{\overline{V_o^2}} = 0.58\ \text{mV}$$

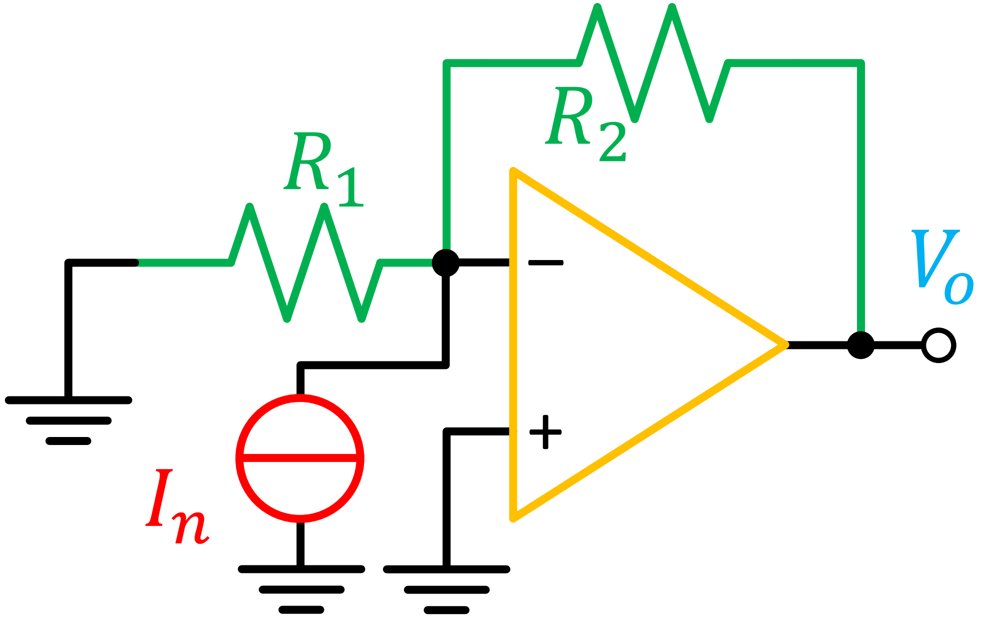
Current source shift theorem



Example: R_2 noise



R_1, R_2, OpAmp current noise



$$S_{I_n} = \frac{4k_BT}{R_1} + \frac{4k_BT}{R_2} + S_I$$

$$\overline{V_o^2} = S_{I_n} R_2^2 \frac{\pi}{2} f_p$$

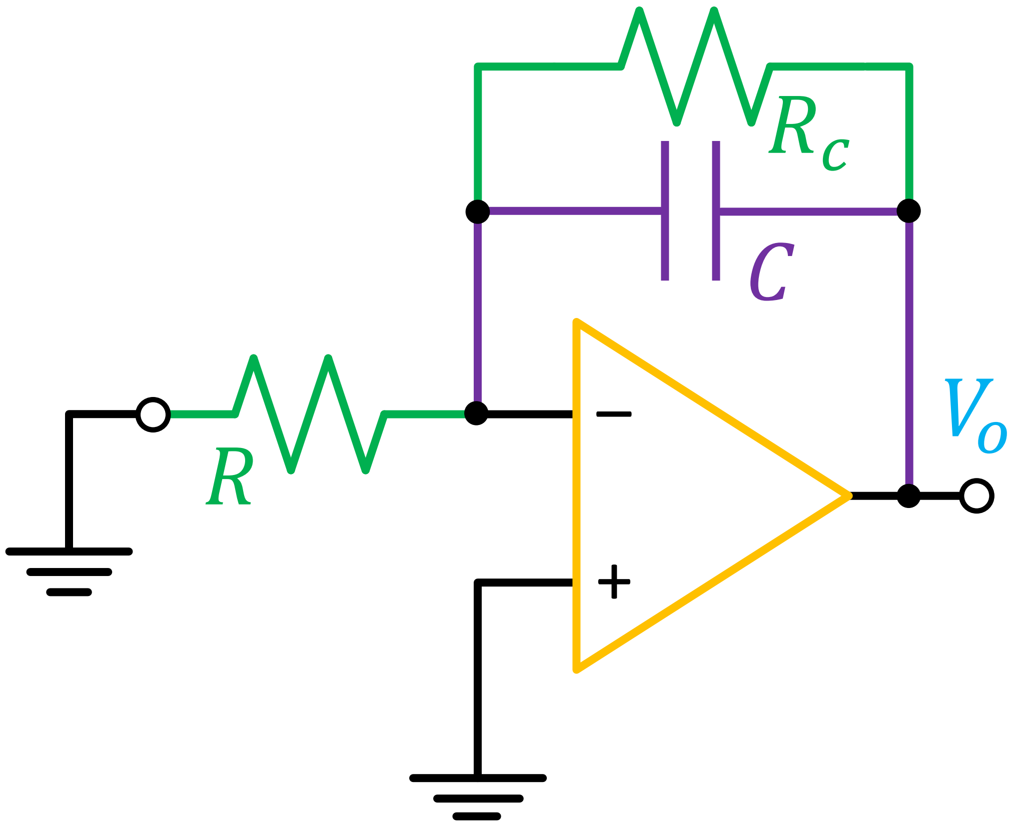
$$\approx 1.83 \times 10^{-7}\ \mathrm{V}^2$$

Many contributions can be grouped into a single noise current source

Outline

- Inverting amplifier
- Integrator
- Differentiator
- Appendix: BP filter and white noise

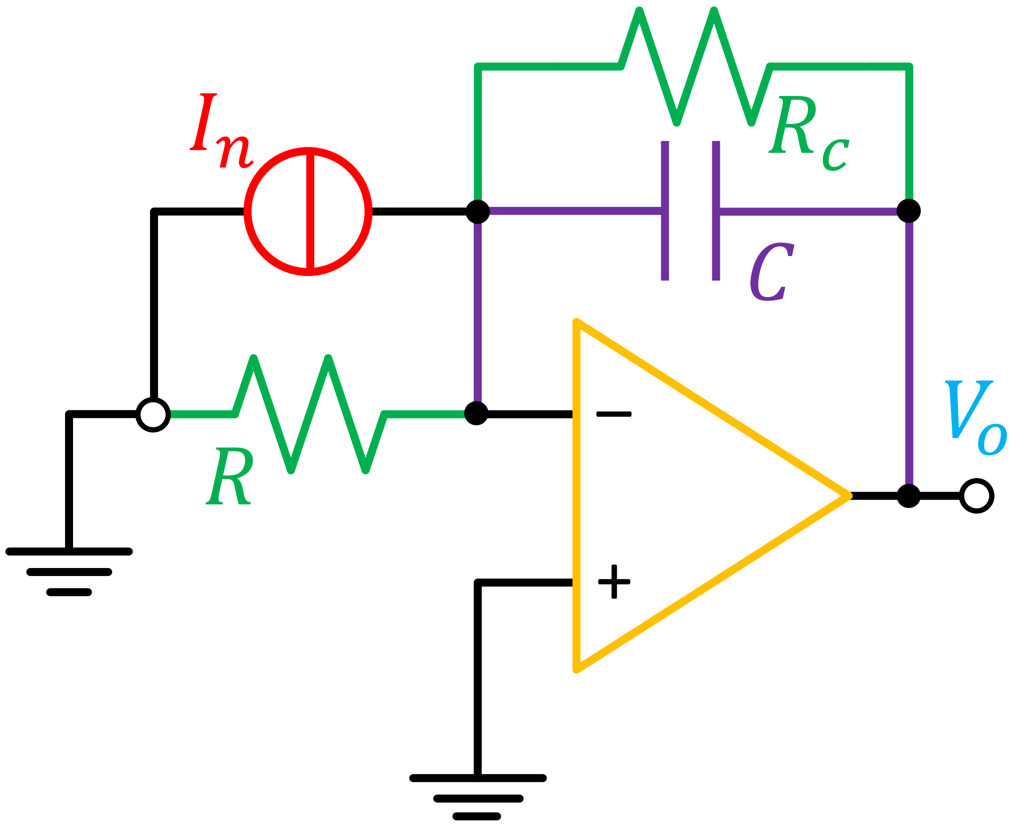
Integrator



- $R = 1\ \text{k}\Omega,\ \ R_c = 1\ \text{M}\Omega$
- $C = 160\ \text{nF}$
- $GBWP = 2\ \text{MHz}$
- $\sqrt{S_V} = 10\ \text{nV}/\sqrt{\text{Hz}}$
- $\sqrt{S_I} = 10\ \text{pA}/\sqrt{\text{Hz}}$

1. Find the total output noise

R, R_c, OpAmp current noise



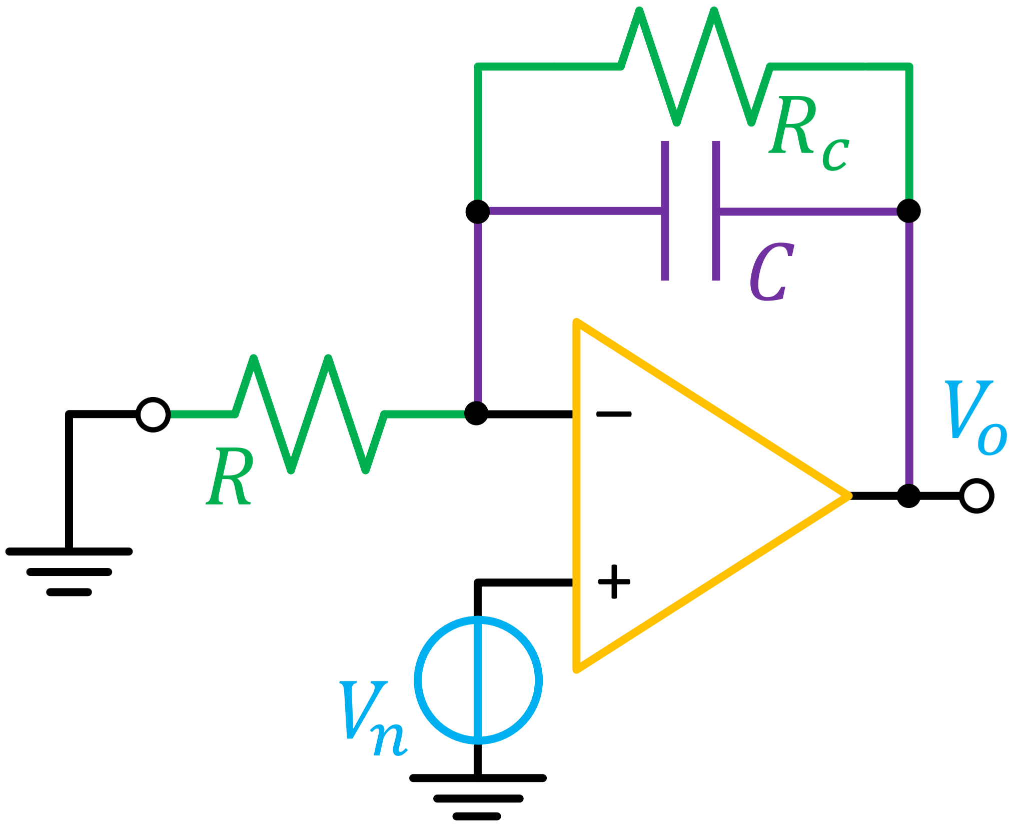
$$S_{I_n} = \frac{4k_BT}{R} + \frac{4k_BT}{R_c} + S_I$$

$$V_o = I_n \frac{R_c}{1 + sCR_c}$$

$$\overline{V_o^2} = S_{I_n} R_c^2 \frac{\pi}{2} \frac{1}{2\pi R_c C}$$

$$\approx 1.82 \times 10^{-10}\ \mathrm{V}^2$$

OpAmp voltage noise



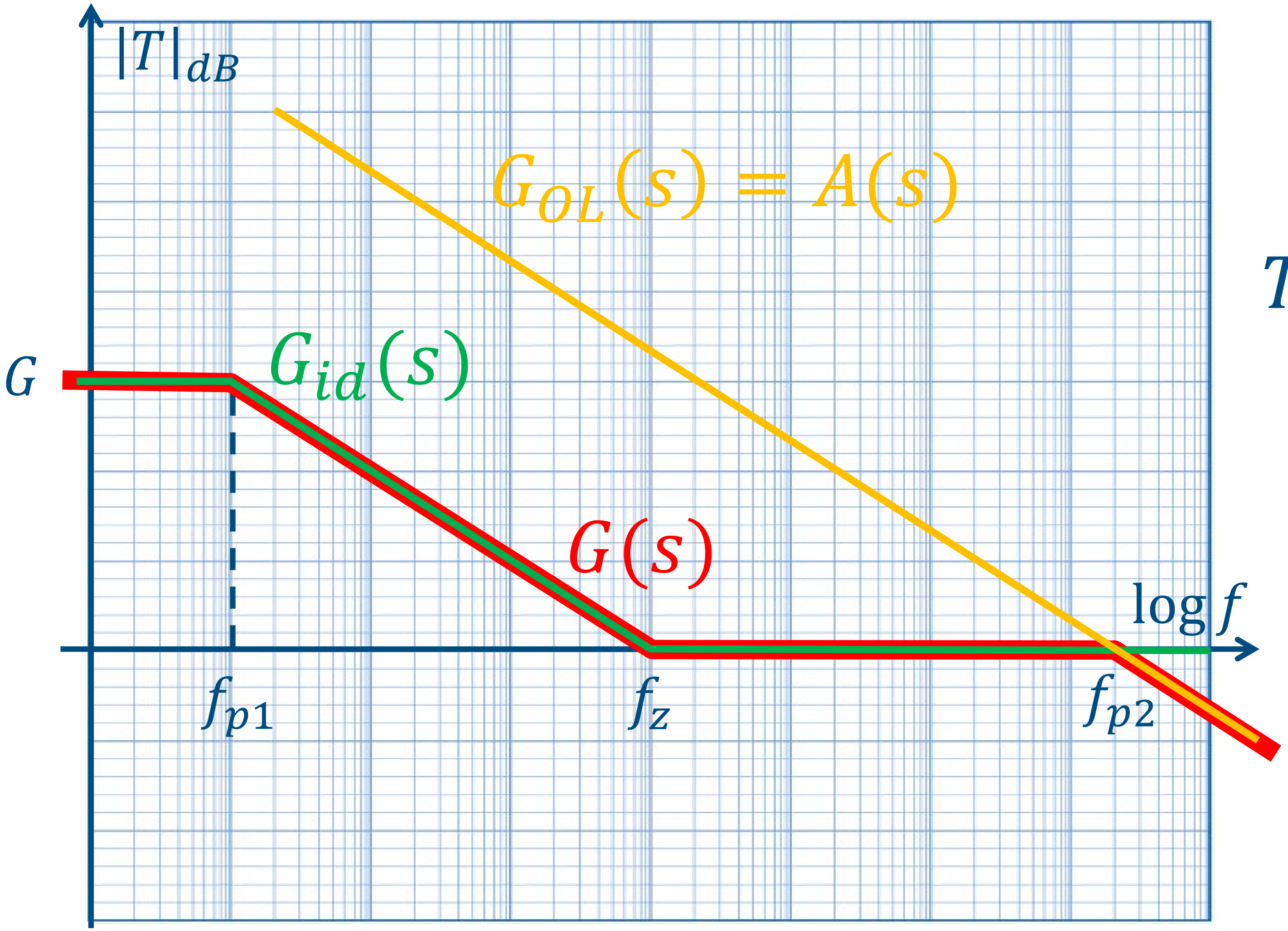
$$V_o = V_n\left(1 + \frac{R_c \parallel 1/sC}{R}\right)$$

$$= V_n \frac{R + R_c}{R}\left(\frac{1 + sC\,R \parallel R_c}{1 + sCR_c}\right)$$

$$\frac{V_o}{V_n} = G\,\frac{1 + s\tau_z}{1 + s\tau_{p1}}$$

$$G \approx 1000, f_{p1} = 1\ \text{Hz}, f_z = 1\ \text{kHz}$$

Closed-loop gain



$$T(s) = \frac{G(1 + s\tau_z)}{(1 + s\tau_{p1})(1 + s\tau_{p2})}$$

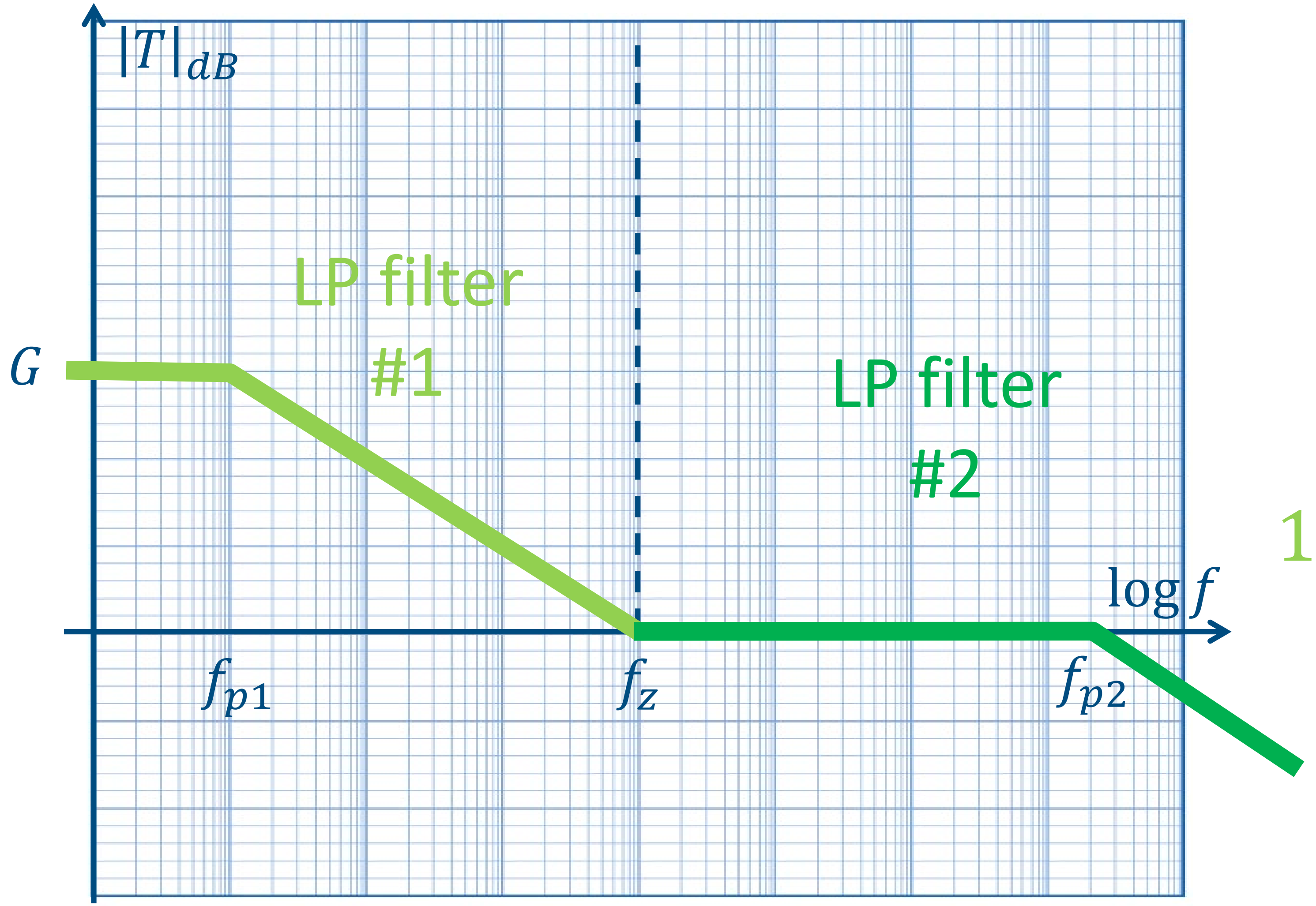
Output noise – full calculations

$$\overline{V_o^2} = \int |T|^2 df = S_V G^2 \int \frac{(1+\omega^2\tau_z^2)}{(1+\omega^2\tau_{p1}^2)(1+\omega^2\tau_{p2}^2)} df$$

$$\int \frac{1+\omega^2\tau_z^2}{(1+\omega^2\tau_{p1}^2)(1+\omega^2\tau_{p2}^2)} df = \int \left(\frac{A}{1+\omega^2\tau_{p1}^2} + \frac{B}{1+\omega^2\tau_{p2}^2} \right) df$$

$$\begin{cases} A + B = 1 \\ A\tau_{p2}^2 + B\tau_{p1}^2 = \tau_z^2 \end{cases} \Rightarrow \begin{aligned} A &= \frac{\tau_{p1}^2 - \tau_z^2}{\tau_{p1}^2 - \tau_{p2}^2} \approx 1 \\ B &= \frac{\tau_z^2 - \tau_{p2}^2}{\tau_{p1}^2 - \tau_{p2}^2} \approx \left(\frac{\tau_z}{\tau_{p1}}\right)^2 \end{aligned}$$

Piecewise linear approximation



$$\int |T|^2 df \approx$$

$$G^2 \frac{\pi}{2} f_{p1} + \frac{\pi}{2} f_{p2} =$$

$$1.57 \times 10^6 + 3.14 \times 10^6$$

Output noise

- The OpAmp voltage noise contribution becomes then

$$\overline{V_o^2} = S_V(1.57 + 3.14) \times 10^6 = 4.71 \times 10^{-10}\ \mathrm{V}^2$$

- The total noise is then

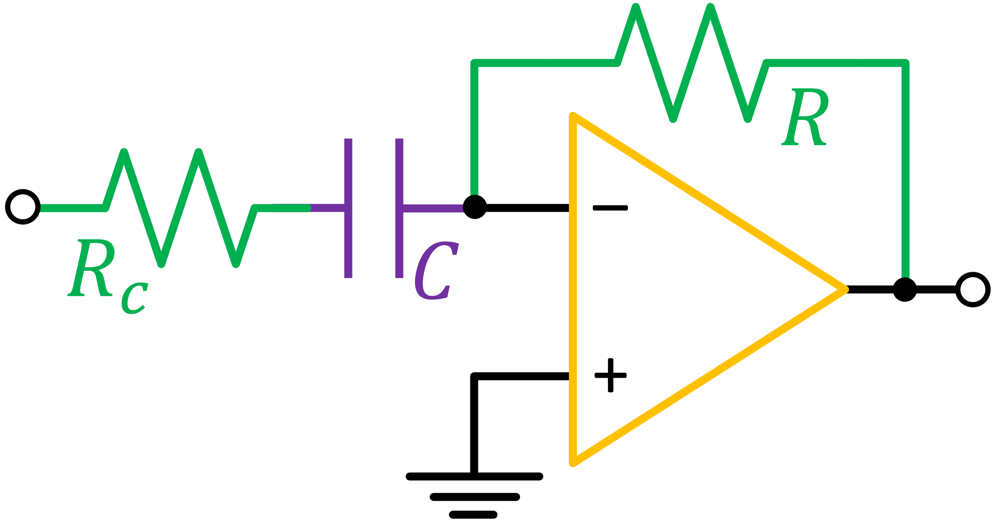
$$\overline{V_o^2} = 1.82 \times 10^{-10} + 4.71 \times 10^{-10} = 6.53 \times 10^{-10}\ \mathrm{V}^2 \Rightarrow$$

$$\sqrt{\overline{V_o^2}} \approx 26\ \mu\mathrm{V}$$

Outline

- Inverting amplifier
- Integrator
- Differentiator
- Appendix: BP filter and white noise

Differentiator



- $R = 14.4\ \mathrm{k\Omega},\ \ R_c = 1.5\ \mathrm{k\Omega}$
- $C = 10\ \mathrm{nF}$
- $GBWP = 10\ \mathrm{MHz}$
- $\sqrt{S_V} = 20\ \mathrm{nV/\sqrt{Hz}}$
- $\sqrt{S_I} = 1\ \mathrm{pA/\sqrt{Hz}}$

1. Find the total output noise

Loop gain calculation

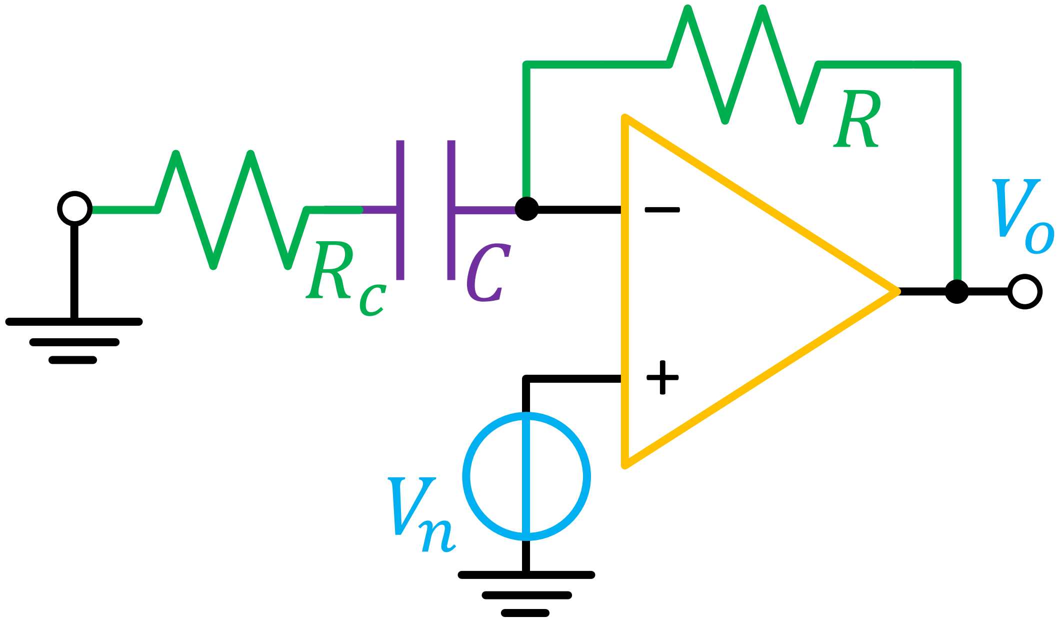
$$G_{loop} = -A(s)\frac{R_c + \frac{1}{sC}}{R + R_c + \frac{1}{sC}} = -A(s)\frac{1 + sCR_c}{1 + sC(R + R_c)}$$

$$f_p = 1\ \text{kHz}, f_z = 10.6\ \text{kHz}$$

Beyond the singularities, we have

$$G_{loop} \approx -\frac{A_0}{s\tau}\frac{R_c}{R + R_c} \Rightarrow f_{0dB} = GBWP\frac{R_c}{R + R_c} = 0.94\ \text{MHz}$$

OpAmp voltage noise



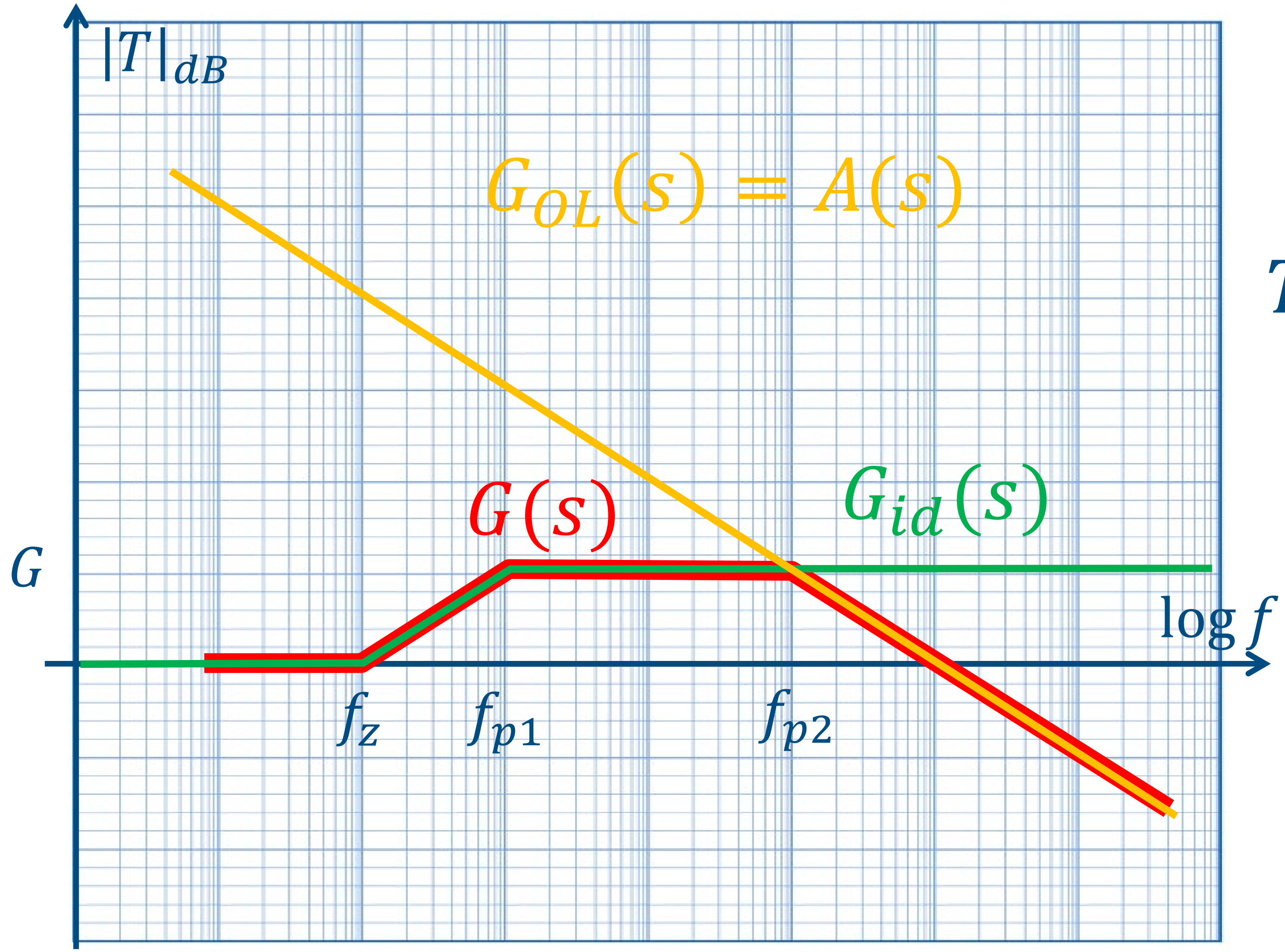
$$V_o = V_n\left(1 + \frac{R}{R_c + 1/sC}\right)$$

$$= V_n \frac{1 + sC(R + R_c)}{1 + sCR_c}$$

$$\frac{V_o}{V_n} = \frac{1 + s\tau_z}{1 + s\tau_{p1}}$$

$$f_{p1} = 10.6\ \text{kHz}, f_z = 1\ \text{kHz}$$

Closed-loop gain



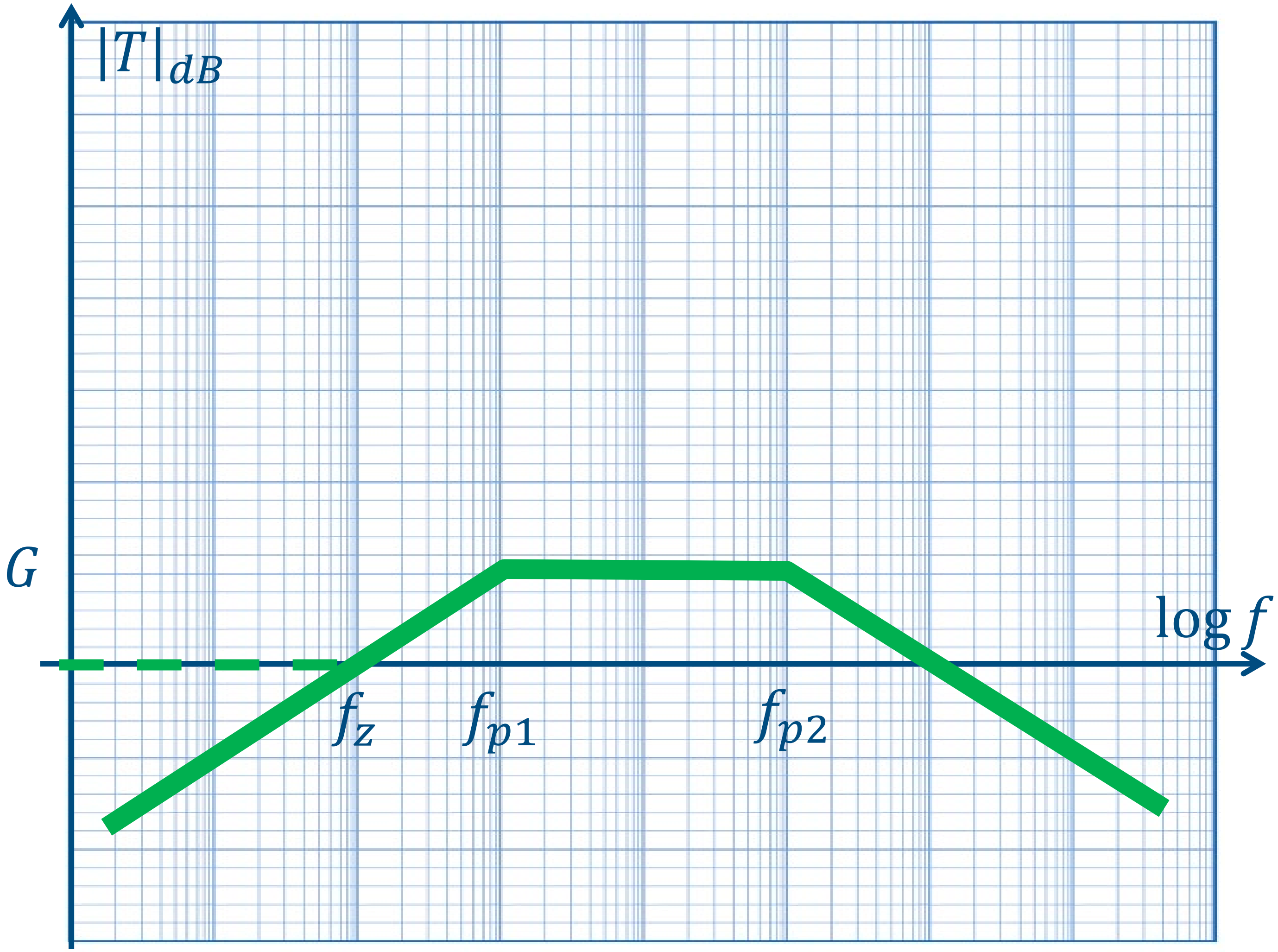
$$T(s) = \frac{1 + s\tau_z}{(1 + s\tau_{p1})(1 + s\tau_{p2})}$$

Output noise – full calculations

$$\overline{V_o^2} = \int |T|^2 df = \int \left(\frac{A}{1+\omega^2\tau_{p1}^2} + \frac{B}{1+\omega^2\tau_{p2}^2} \right) df$$

$$\begin{cases} A + B = 1 \\ A\tau_{p2}^2 + B\tau_{p1}^2 = \tau_z^2 \end{cases} \Rightarrow \begin{matrix} A = \dfrac{\tau_{p1}^2 - \tau_z^2}{\tau_{p1}^2 - \tau_{p2}^2} \approx -\left(\dfrac{\tau_z}{\tau_{p1}}\right)^2 = -G^2 \\ B = \dfrac{\tau_z^2 - \tau_{p2}^2}{\tau_{p1}^2 - \tau_{p2}^2} \approx \left(\dfrac{\tau_z}{\tau_{p1}}\right)^2 = G^2 \end{matrix}$$

Piecewise linear approximation



$$\int |T|^2 df \approx$$

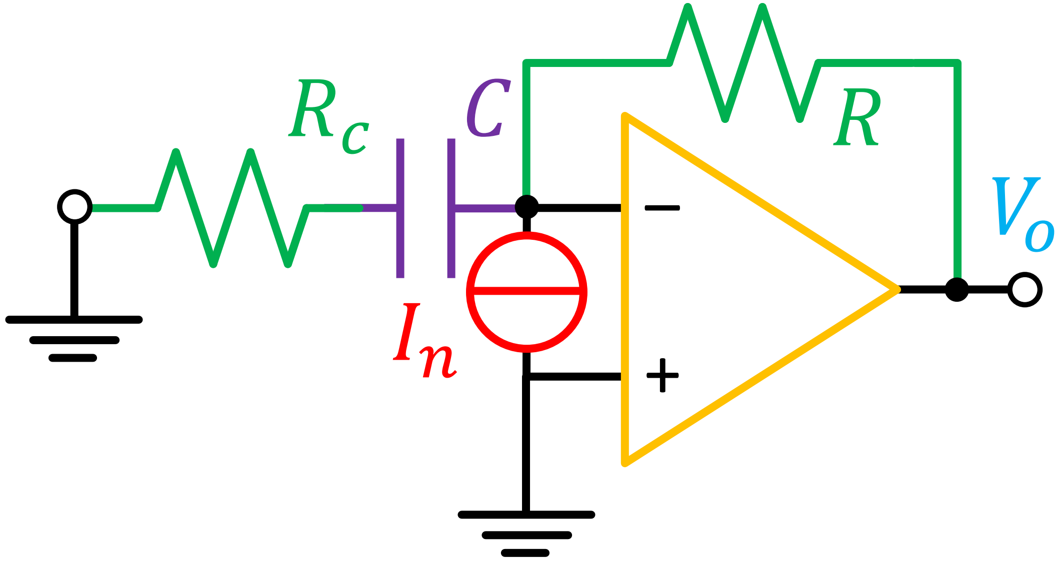
$$G^2 \frac{\pi}{2}(f_{p2} - f_{p1}) =$$

$$1.64 \times 10^8$$

$$\overline{V_o^2} = 1.64 \times 10^8\, S_V$$

$$= 6.56 \times 10^{-8}\ \mathrm{V}^2$$

OpAmp + R current noise



$$S_{I_n} = S_I + \frac{4k_BT}{R}$$

$$V_o = I_nR$$

$$\overline{V_o^2} = S_{I_n}R^2\frac{\pi}{2}f_{0dB}$$

$$= 3.06 \times 10^{-10}\ \mathrm{V}^2$$

R_c noise

Transfer function is the same as the input

$$V_o = V_n \frac{sCR}{1 + sCR_c}$$

With the addition of the pole at f_{0dB} we have a BP filter with mid-band gain equal to $R/R_c = 9.6$

$$\overline{V_o^2} = S_{R_c}\left(\frac{R}{R_c}\right)^2 \frac{\pi}{2}(f_{0dB} - f_{p1}) = 2.14 \times 10^{-9}\ \mathrm{V}^2$$

Output noise

The total noise is then

$$\overline{V_o^2} = 6.56 \times 10^{-8} + 3.06 \times 10^{-10} + 2.14 \times 10^{-9}$$

$$= 6.61 \times 10^{-8}\ \mathrm{V}^2 \Rightarrow \sqrt{\overline{V_o^2}} \approx 261\ \mu\mathrm{V}$$

Outline

- Inverting amplifier
- Integrator
- Differentiator
- Appendix: BP filter and white noise

BP filter and white noise

$$H(s) = \frac{s\tau_L}{(1+s\tau_L)(1+s\tau_H)}$$

$$\overline{V_o^2} = S_V \int_0^\infty \frac{(\omega\tau_L)^2}{(1+\omega^2\tau_L^2)(1+\omega^2\tau_H^2)} \frac{d\omega}{2\pi}$$

$$= \frac{S_V}{2\pi} \frac{\tau_L^2}{\tau_H^2 - \tau_L^2} \int_0^\infty \left(\frac{1}{1+\omega^2\tau_L^2} - \frac{1}{1+\omega^2\tau_H^2} \right) d\omega = \frac{S_V}{2\pi} \frac{\tau_L^2}{\tau_H^2 - \tau_L^2} \left(\frac{\pi}{2\tau_L} - \frac{\pi}{2\tau_H} \right)$$

$$= \frac{S_V}{4} \frac{\tau_L}{\tau_H(\tau_L + \tau_H)} = \frac{S_V}{4} \frac{1}{\tau_H(1+\tau_H/\tau_L)} \approx \frac{S_V}{4} \frac{1}{\tau_H} \left(1 - \frac{\tau_H}{\tau_L} \right) = \frac{S_V}{4} \left(\frac{1}{\tau_H} - \frac{1}{\tau_L} \right)$$

$$= S_V \frac{\pi}{2} (f_H - f_L)$$

-3dB bandwidth and noise

- The maximum of $|H(j\omega)| = \frac{\omega\tau_L}{|1-\omega^2\tau_L\tau_H+j\omega(\tau_L+\tau_H)|}$ is at

$$\omega_p = \frac{1}{\sqrt{\tau_L\tau_H}} \qquad \left|H\left(j\omega_p\right)\right| = \frac{\tau_L}{\tau_L+\tau_H}$$

- The -3dB BW is given by

$$\frac{\omega\tau_L}{|1-\omega^2\tau_L\tau_H+j\omega(\tau_L+\tau_H)|} = \frac{1}{\sqrt{2}}\frac{\tau_L}{\tau_L+\tau_H}$$

$$\left(1-\omega^2\tau_L\tau_H\right)^2+\omega^2(\tau_L+\tau_H)^2 = 2\omega^2(\tau_L+\tau_H)^2 \Rightarrow 1-\omega^2\tau_L\tau_H = \pm\omega(\tau_L+\tau_H) \Rightarrow \omega^2 \pm \omega\frac{\tau_L+\tau_H}{\tau_L\tau_H} - \frac{1}{\tau_L\tau_H} = 0$$

- We pick only the positive solutions:

$$\omega_L = -\frac{\tau_L+\tau_H}{2\tau_L\tau_H} + \sqrt{\left(\frac{\tau_L+\tau_H}{2\tau_L\tau_H}\right)^2+\frac{1}{\tau_L\tau_H}} \qquad \omega_H = \frac{\tau_L+\tau_H}{2\tau_L\tau_H} + \sqrt{\left(\frac{\tau_L+\tau_H}{2\tau_L\tau_H}\right)^2+\frac{1}{\tau_L\tau_H}}$$

$$BW = \frac{\omega_H-\omega_L}{2\pi} = \frac{1}{2\pi}\frac{\tau_L+\tau_H}{\tau_L\tau_H} \qquad \overline{V_o^2} = S_V\left|H\left(f_p\right)\right|^2\frac{\pi}{2}BW = S_V\left(\frac{\tau_L}{\tau_L+\tau_H}\right)^2\frac{\tau_L+\tau_H}{4\tau_L\tau_H} = \frac{S_V}{4}\frac{\tau_L}{\tau_H(\tau_L+\tau_H)}$$

General case: the quality factor

- The second-order polynomials describing two poles (or zeros) can be written as

$$H(s) = 1 + \frac{s}{\omega_p Q} + \frac{s^2}{\omega_p^2}$$

- Q is the quality factor of the poles:
  - $0 < Q < 0.5 \Rightarrow$ real and separated poles
  - $Q = 0.5 \Rightarrow$ real and coincident poles at ω_p
  - $Q > 0.5 \Rightarrow$ complex conjugated poles at ω_p

Quality factor, BW and noise

- Consider a BP filter

$$H(s) = \frac{s/\omega_p}{1 + \frac{s}{\omega_p Q} + \frac{s^2}{\omega_p^2}}$$

- The maximum of $|H(j\omega)|$ is at $\omega = \omega_p$, where $\left|H\left(j\omega_p\right)\right| = Q$
- The -3dB bandwidth (for both real and complex poles) is

$$BW = \frac{f_p}{Q}$$

- The output noise is

$$\overline{V_o^2} = S_V \frac{\pi}{2} BW \left|H\left(j\omega_p\right)\right|^2 = S_V \frac{\pi}{2} BW Q^2 = S_V \frac{\pi}{2} Q f_p$$