Electronics – 96032

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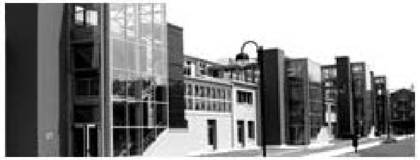    

Optimum filter and discrete-time filters

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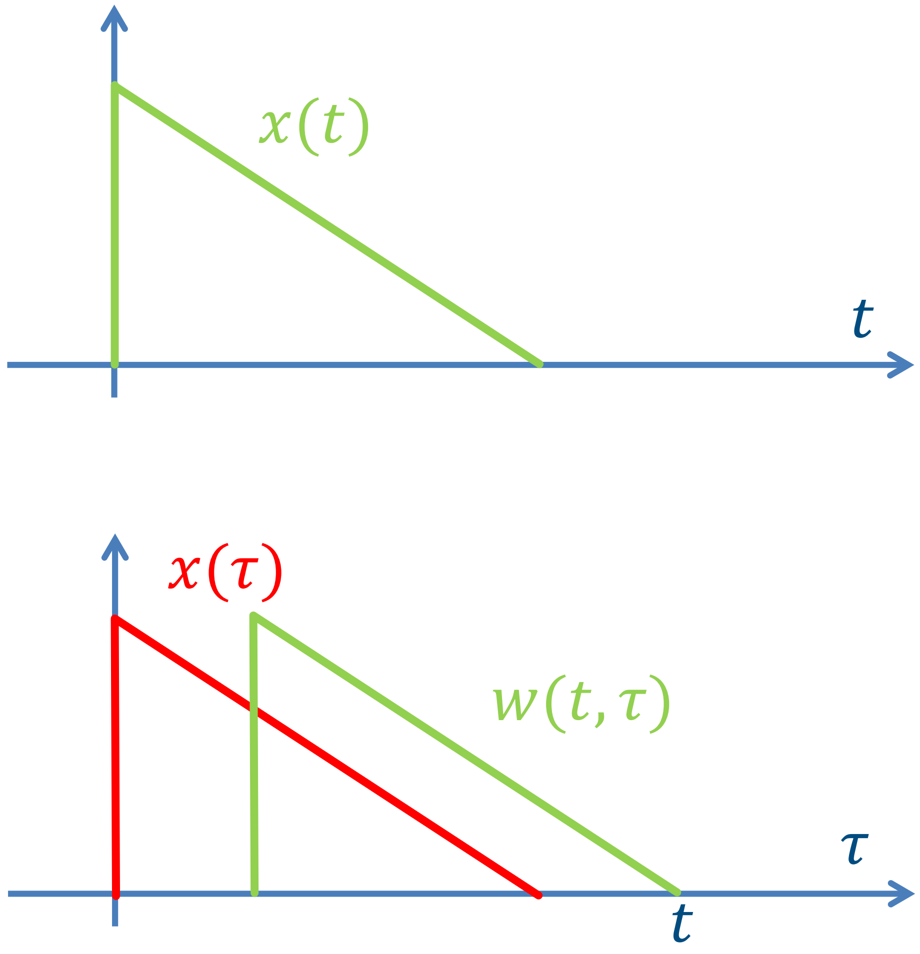
Disclaimer

Slides are supplementary material and are NOT a replacement for textbooks and/or lecture notes

Outline

- Optimum filter
  - Review
  - Problems
- Discrete-time filters
  - Review
  - Problems

Optimum filter review



- Weighting function for the WN case

$$w(t,\tau) \propto x(\tau)$$

- Output signal

$$y(t) = \int w^2(t,\tau)d\tau$$

- Output noise (bilateral PSD λ)

$$\overline{n_y^2} = \lambda \int w^2(t,\tau)d\tau$$

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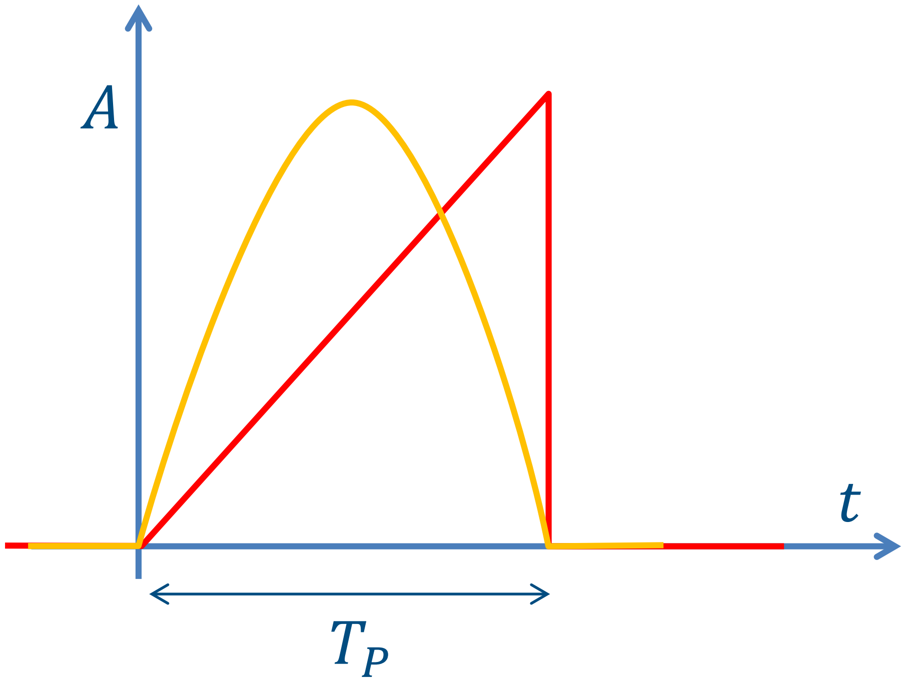
Problem: triangular/sinusoidal signals



1. Compute the maximum S/N
2. Compare the results with what obtained previously with GIs

Triangular signal

- Weighting function

$$w(t,\tau) = Gx(\tau) = GA\frac{\tau}{T_P}$$

- Output signal and noise

$$y(t) = \frac{GA^2}{T_P^2}\int_0^{T_P}\tau^2 d\tau = \frac{GA^2 T_P}{3} \qquad \overline{n_y^2} = \lambda\int w^2(t,\tau)d\tau = \lambda\left(\frac{GA}{T_P}\right)^2\int_0^{T_P}\tau^2 d\tau = \lambda(GA)^2\frac{T_P}{3}$$

- Output S/N

$$\frac{S}{N} = \frac{AT_P}{3\sqrt{\lambda\frac{T_P}{3}}} = A\sqrt{\frac{T_P}{\lambda}}\frac{1}{\sqrt{3}}$$

Comparison

- In the GI case we have

$$\frac{S}{N} = A\sqrt{\frac{T_P}{2\lambda}\frac{4}{3\sqrt{3}}} = A\sqrt{\frac{T_P}{\lambda}\frac{4}{3\sqrt{6}}} \Rightarrow \frac{(S/N)_{opt}}{(S/N)_{GI}} = \frac{3\sqrt{2}}{4} = 1.06$$

- This is a common result: the improvement you get with an OF is usually small with respect to a WF that «approximates» its shape

Sinusoidal signal

- OF result

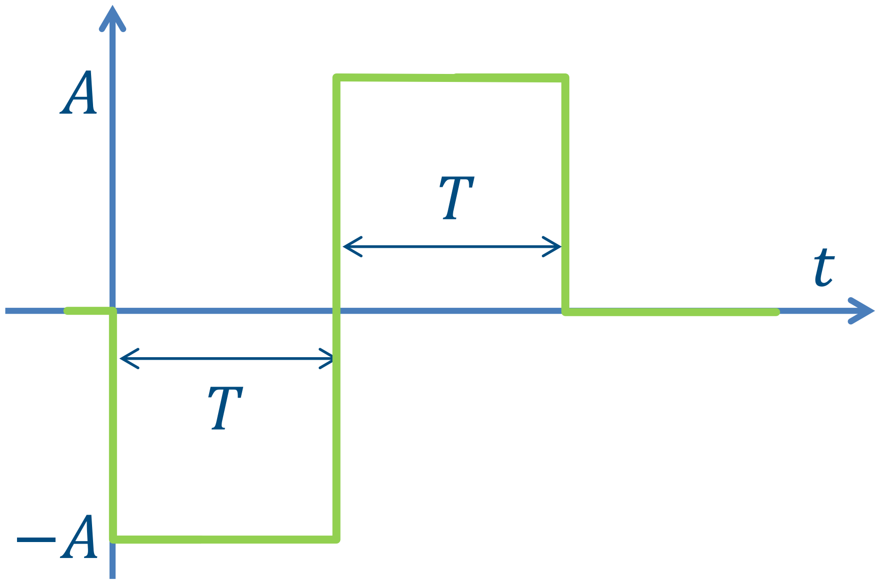
$$\left(\frac{S}{N}\right)_{opt} = \frac{A}{\sqrt{\lambda}}\sqrt{\int x^2(t)dt} = \frac{A}{\sqrt{\lambda}}\sqrt{\int_0^{T_P} \sin^2 \omega t\, dt} = \frac{A}{\sqrt{\lambda}}\sqrt{\frac{T_P}{2}}$$

- GI result

$$\left(\frac{S}{N}\right)_{GI} = A\sqrt{\frac{2T_P}{\lambda}\frac{\sin 1.17}{\sqrt{1.17\pi}}}$$

$$\frac{(S/N)_{opt}}{(S/N)_{GI}} = \frac{\sqrt{1.17\,\pi}}{2\sin 1.17} = 1.04$$

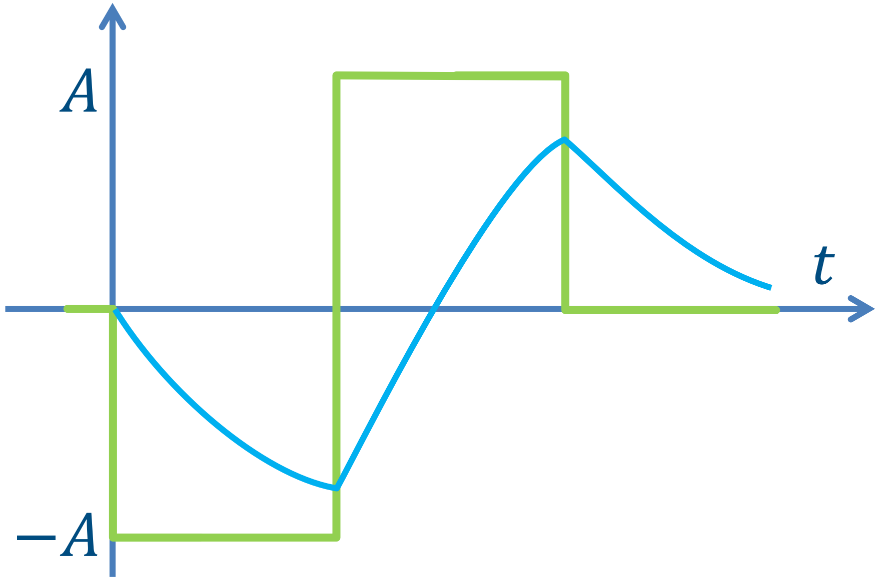
Problem: square wave signal + WN



Current signal affected by WN with PSD λ_I (bilateral)

1. Design a simple LPF and compute S/N
2. Plot the output signal of the OF vs time and compute S/N
3. Repeat #2 when the signal is affected by shot noise

LPF

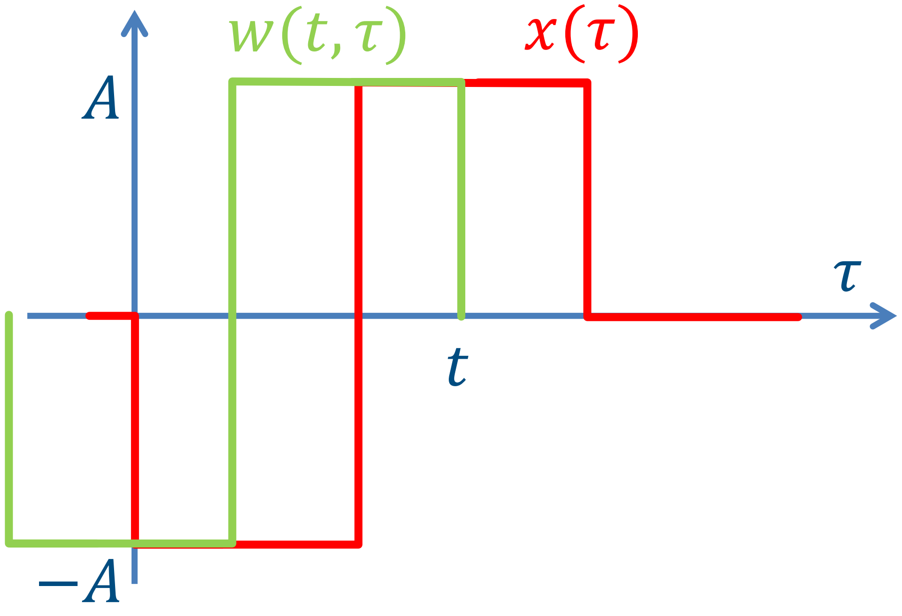


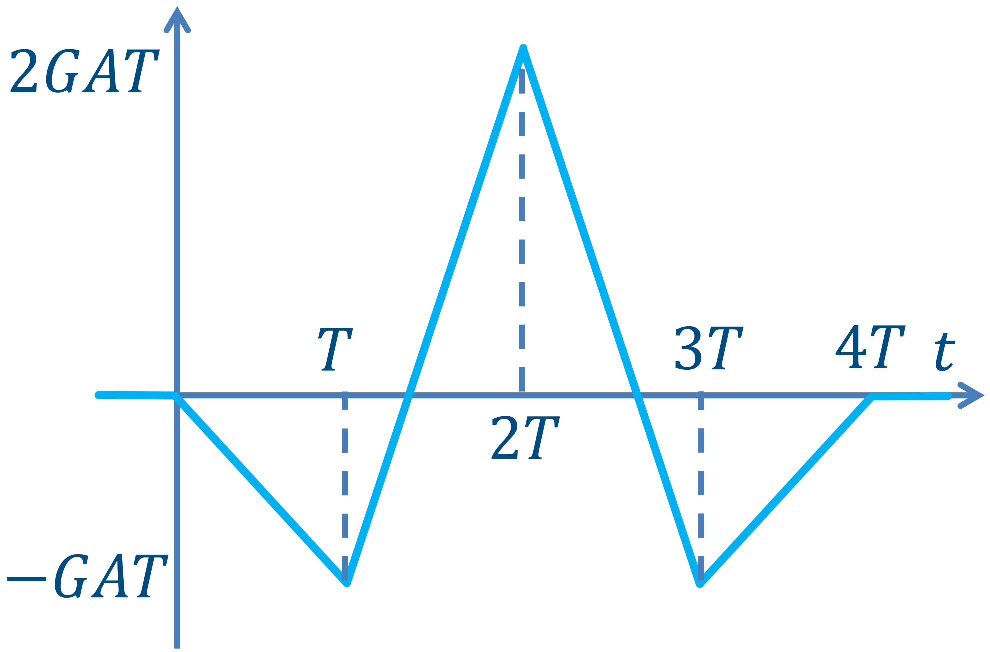
- We sample the output at the highest (absolute) value, i.e., at $t = T$
- The result is

$$\left(\frac{S}{N}\right)_{LPF} = A\frac{1 - e^{-T/RC}}{\sqrt{\frac{\lambda_I}{2RC}}}$$

- We can pick $RC = T$ (optimum value is $\approx 0.8T$)

OF output signal and S/N





$$\left(\frac{S}{N}\right)_{OF} = \frac{A}{\sqrt{\lambda_I}}\sqrt{\int x^2(t)dt} = \frac{A}{\sqrt{\lambda_I}}\sqrt{2T}$$

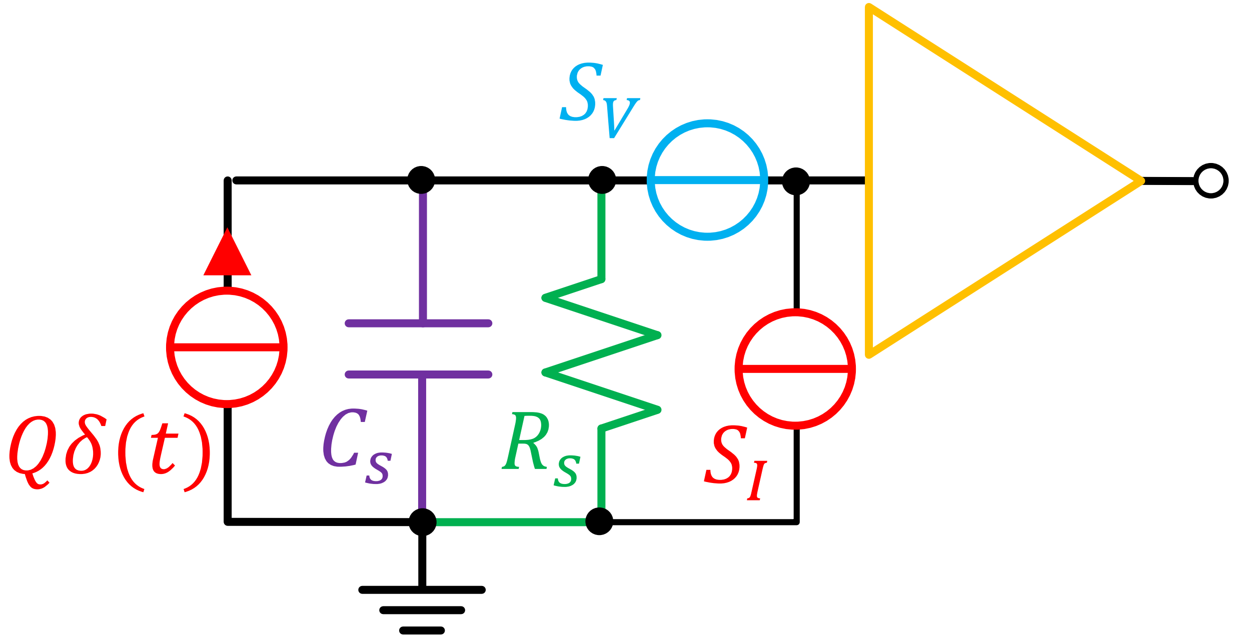
OF and shot noise

- For the case of non-stationary white noise the OF WF is

$$w(t,\tau) = G\frac{x(\tau)}{\lambda(\tau)}$$

- In the shot noise case, $\lambda(\tau) = qx(\tau)$ is proportional to the current signal $\Rightarrow$ $w(t,\tau) = \text{constant}$ (gated integrator)
- The noise PSD does not depend on the sign of $x(\tau)$. In this case $\lambda(\tau) = q|x(\tau)|$ is constant and the OF is the same as the previous one

Problem: sensor



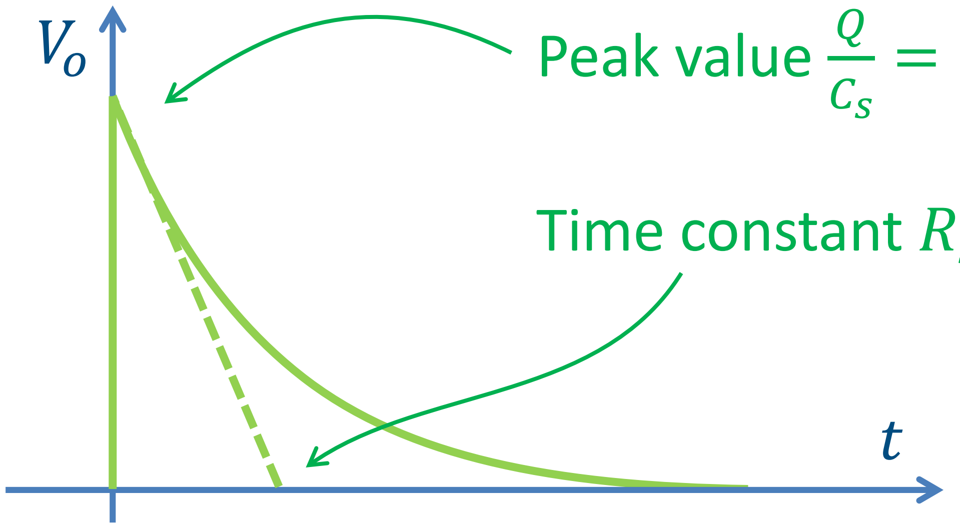
- $Q = 1$ fC
- $C_s = 5$ pF, $R_s = 1$ MΩ
- $S_V = 10$ nV/$\sqrt{\text{Hz}}$
- $S_I = 20$ fA/$\sqrt{\text{Hz}}$

1. Plot the output signal and noise PSD
2. Compute the OF WF

Output signal

In the frequency domain $I(s) = Q$

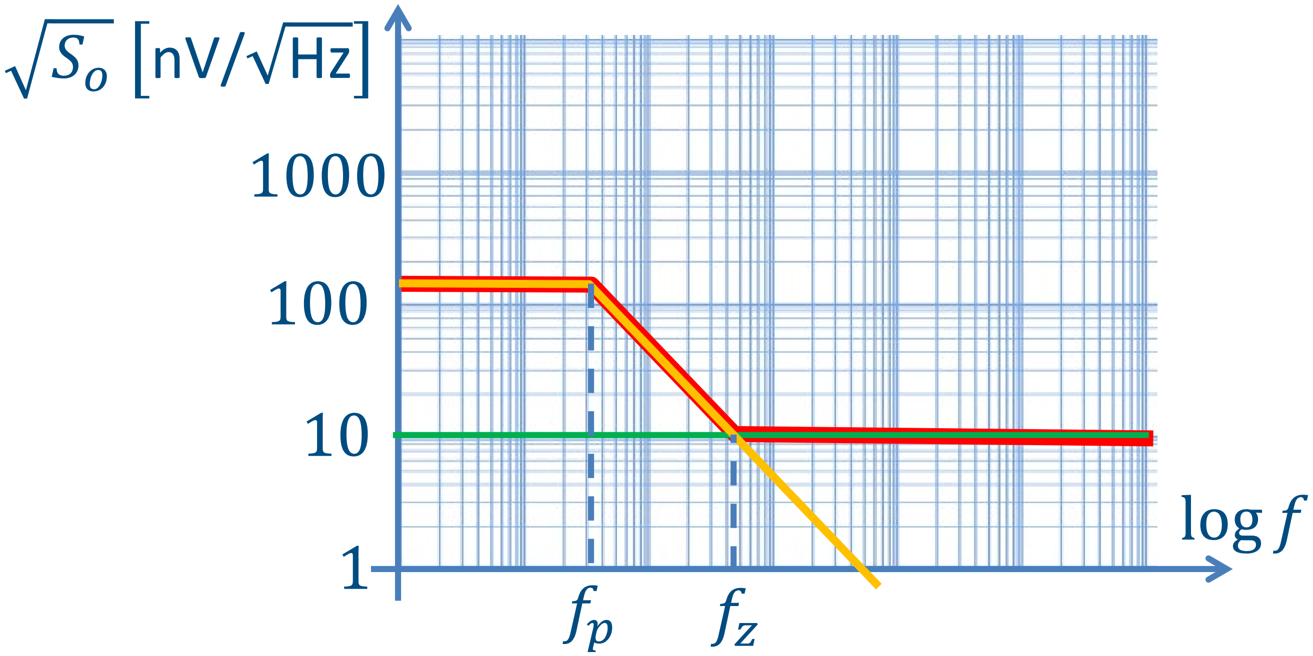
$$V_o(s) = I\ R_s \parallel Z_C = Q \frac{R_s}{1 + sC_sR_s} \Rightarrow V_o(t) = \frac{Q}{C_s} e^{-\frac{t}{R_sC_s}}$$



Output noise

$$S_o = S_V + \left(S_I + \frac{4k_BT}{R_s}\right)\left|\frac{R_s}{1 + sC_sR_s}\right|^2 = S_V + \frac{S_IR_s^2 + 4k_BTR_s}{|1 + sC_sR_s|^2}$$

(S_V: 10^{-16}; $S_IR_s^2$: 4×10^{-16}; $4k_BTR_s$: 1.66×10^{-14})

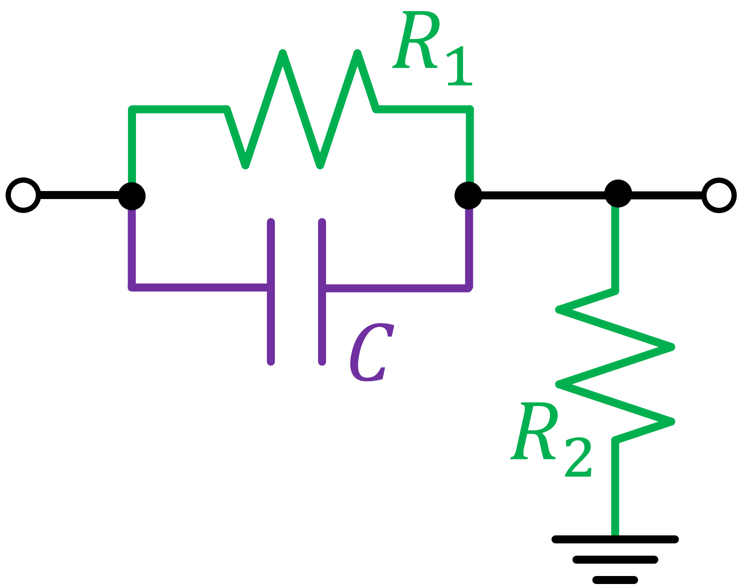


$$f_p = \frac{1}{2\pi C_sR_s} = 31.8\ \text{kHz}$$

$$f_z = f_p\sqrt{\frac{S_V + S_IR_s^2 + 4k_BTR_s}{S_V}} \approx 416\ \text{kHz}$$

Whitening filter

$$S_o|H_w|^2 = K \Rightarrow |H_w|^2 = \frac{K}{S_o} = \frac{K'}{\left|\frac{1+s\tau_z}{1+s\tau_p}\right|^2} = K'\left|\frac{1+s\tau_p}{1+s\tau_z}\right|^2$$



$$= \frac{1}{2\pi f_p} = 5\ \mu s$$

$$H_w(s) = \frac{R_2}{R_1+R_2}\frac{1+sCR_1}{1+sC(R_1 \parallel R_2)}$$

$$= \frac{1}{2\pi f_z} \approx 383\ \text{ns}$$

Whitening filter output

- Output noise

$$S_W = S_V$$

- Output signal

$$V_W = V_o(s)H_w(s) = Q\frac{R_s}{1+sC_sR_s}\frac{R_2}{R_1+R_2}\frac{1+sCR_1}{1+sC(R_1 \parallel R_2)}$$

$$= Q\frac{R_2}{R_1+R_2}\frac{R_s}{1+sC(R_1 \parallel R_2)} \Rightarrow V_o(t) = \frac{QR_s}{CR_1}e^{-\frac{t}{\tau_z}} = \frac{Q}{C_s}e^{-\frac{t}{\tau_z}}$$

Optimum S/N

- The optimum S/N is now

$$\left(\frac{S}{N}\right)_{opt} = \frac{Q}{C_s}\sqrt{\frac{2}{S_W}}\sqrt{\int_0^{\infty} e^{-2t/\tau_z}\,dt} = \frac{Q}{C_s}\sqrt{\frac{\tau_z}{S_W}} = 12.38$$

- The same value could have been obtained from

$$\left(\frac{S}{N}\right)^2_{opt} = A^2 \int \frac{|X(f)|^2}{S_n(f)}\,df$$

Calculations

$$\left(\frac{S}{N}\right)^2_{opt} = Q^2R_s^2 \int \frac{\left|\frac{1}{1+s\tau_p}\right|^2}{\frac{S_0}{2}\left|\frac{1+s\tau_z}{1+s\tau_p}\right|^2} df = \frac{4Q^2R_s^2}{S_0}\int_0^\infty \frac{df}{1+\omega^2\tau_z^2}$$

$$= \frac{Q^2R_s^2}{S_0\tau_z} = \frac{Q^2R_s^2\tau_z}{S_V\tau_p^2} = \frac{Q^2\tau_z}{C_s^2S_V}$$

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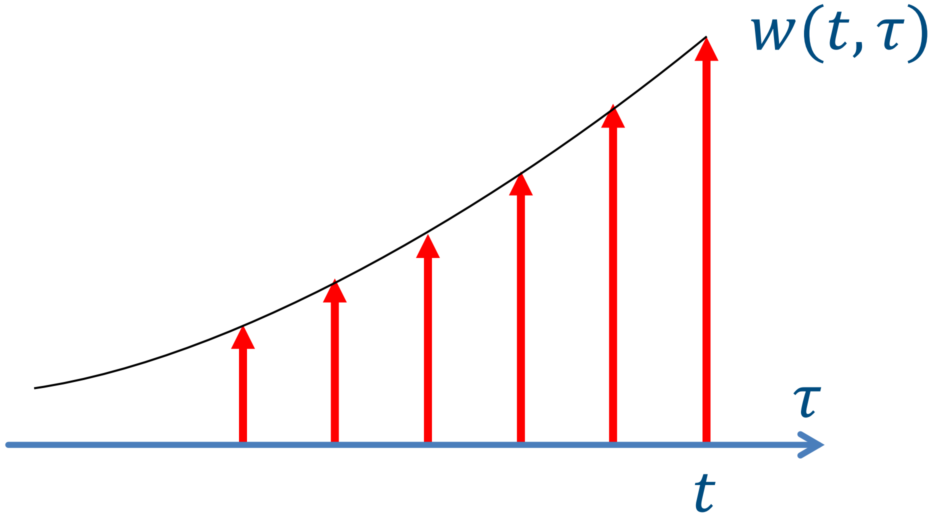
DTF review



- WF is a series of delta functions

$$w(t,\tau) = \sum_{k=0}^{N-1} w_k\, \delta(\tau - (t - kt_s))$$

- Output signal

$$y(t) = \int x(\tau) w(t,\tau) d\tau = \sum_{k=0}^{N-1} w_k\, x(kt_s)$$

- For non-correlated input noise we have

$$\overline{n_y^2} = \overline{n_x^2} \sum_{k=0}^{N-1} w_k^2$$

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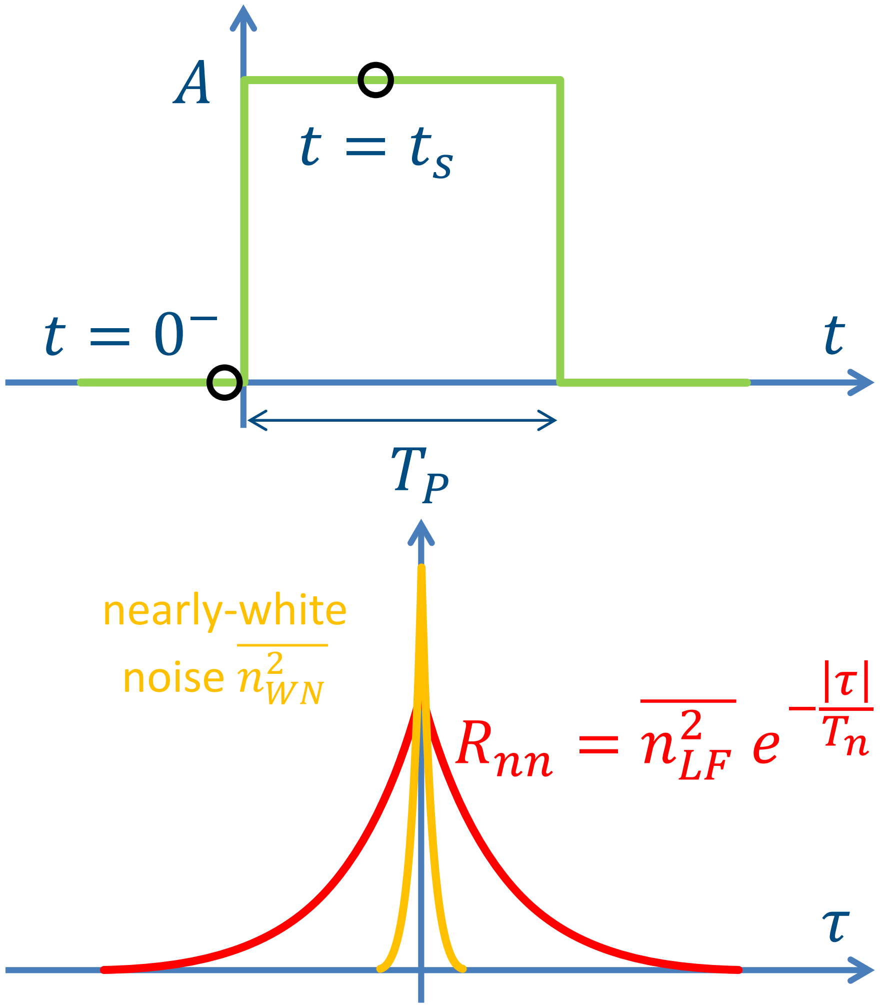
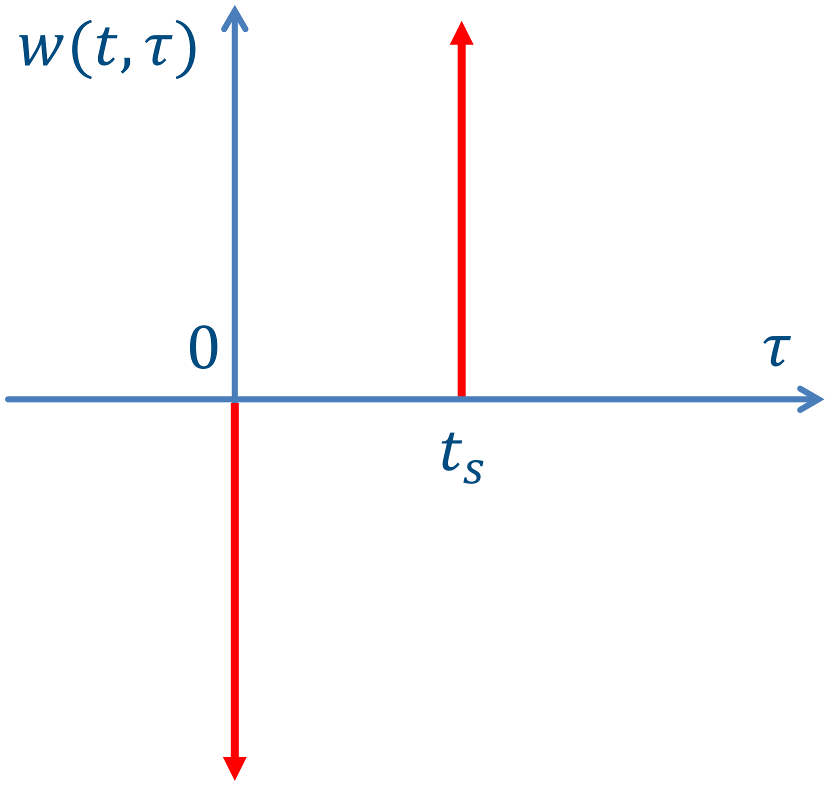
Problem: rectangular pulse + noise



Rectangular pulse + two noise sources. Output is the difference of two samples

1. Compute the WF in the time and frequency domains
2. Compute the output noise
3. Propose an improvement of the filter

Weighting function



- The WF of an ideal sampling operation is a delta function. We then have

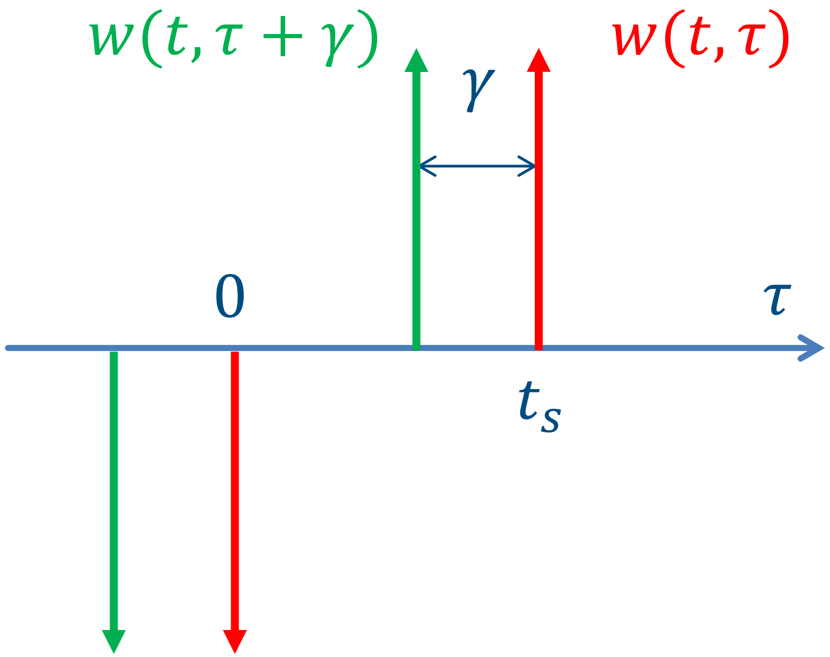
$$w(t,\tau) = \delta(\tau - t_s) - \delta(\tau)$$

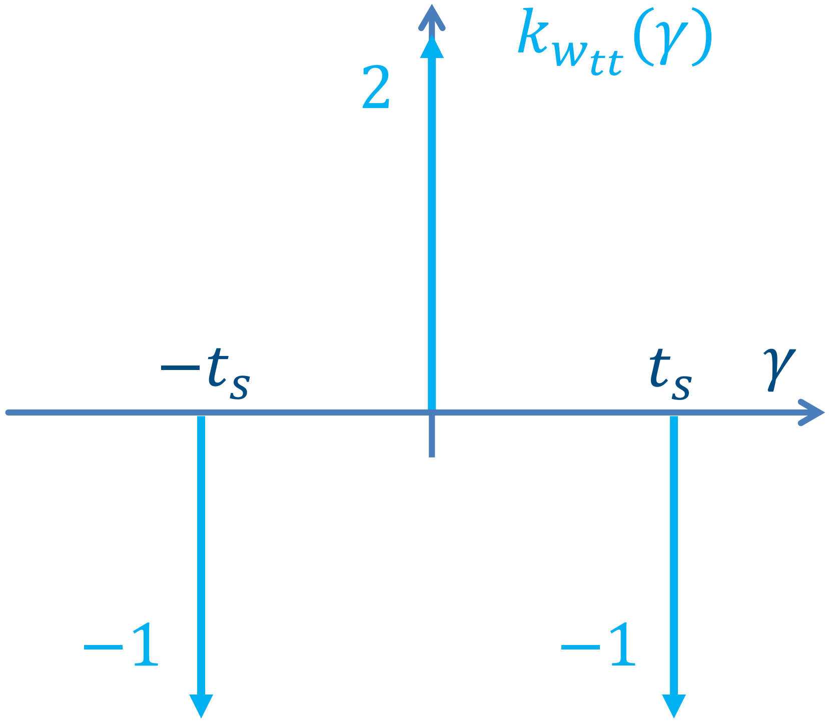
- In the frequency domain

$$W(t,f) = 1 - e^{-j2\pi f t_s}$$

$$|W(t,f)|^2 = 2 - 2\cos 2\pi f t_s$$

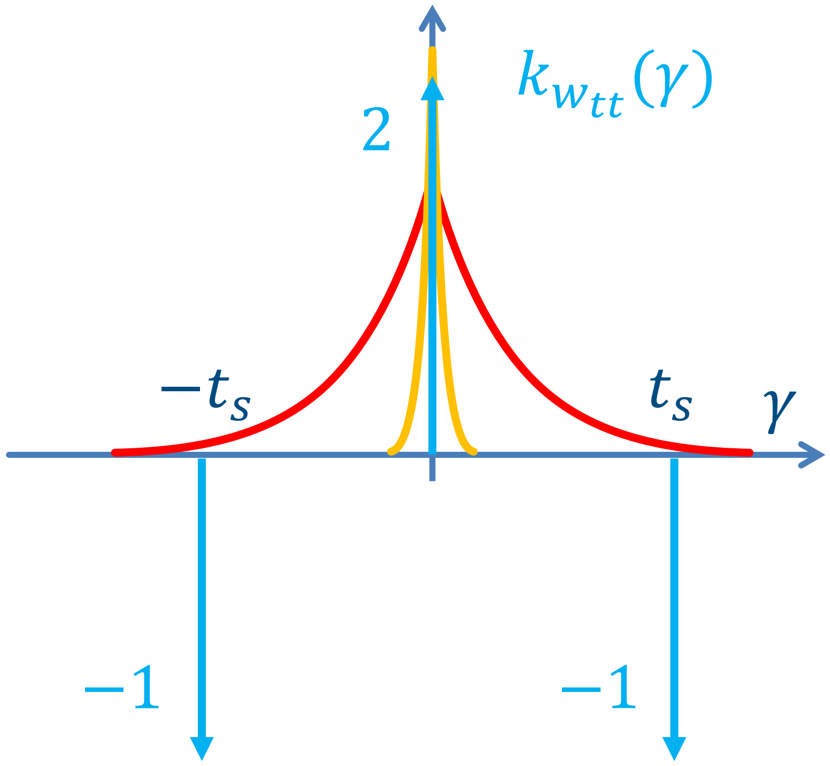
WF time correlation





$$k_{w_{tt}}(\gamma) = 2\delta(\gamma) - \delta(|\gamma| - t_s)$$

Output noise



$$\overline{n_y^2} = \int R_{xx}(\gamma) k_{w_{tt}}(\gamma) d\gamma$$

- WN contribution:

$$\overline{n_y^2} = 2\overline{n_{WN}^2}$$

No improvement!

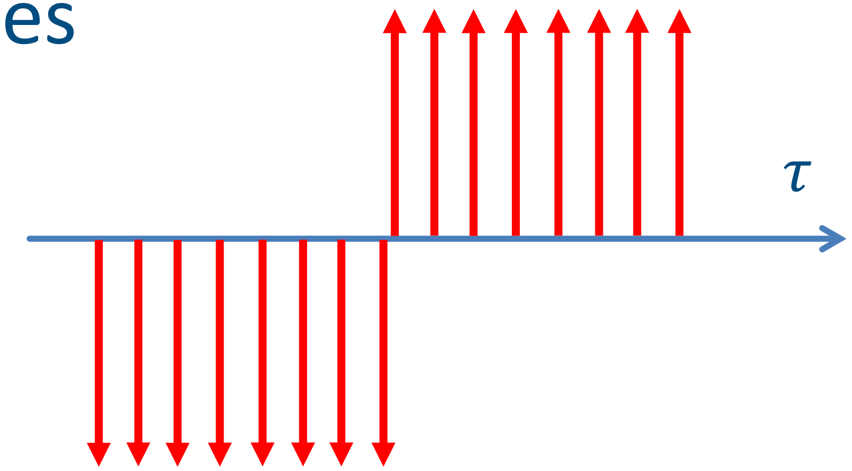
- LF contribution:

$$\overline{n_y^2} = 2\overline{n_{WN}^2}\left(1 - e^{-\frac{t_s}{T_n}}\right)$$

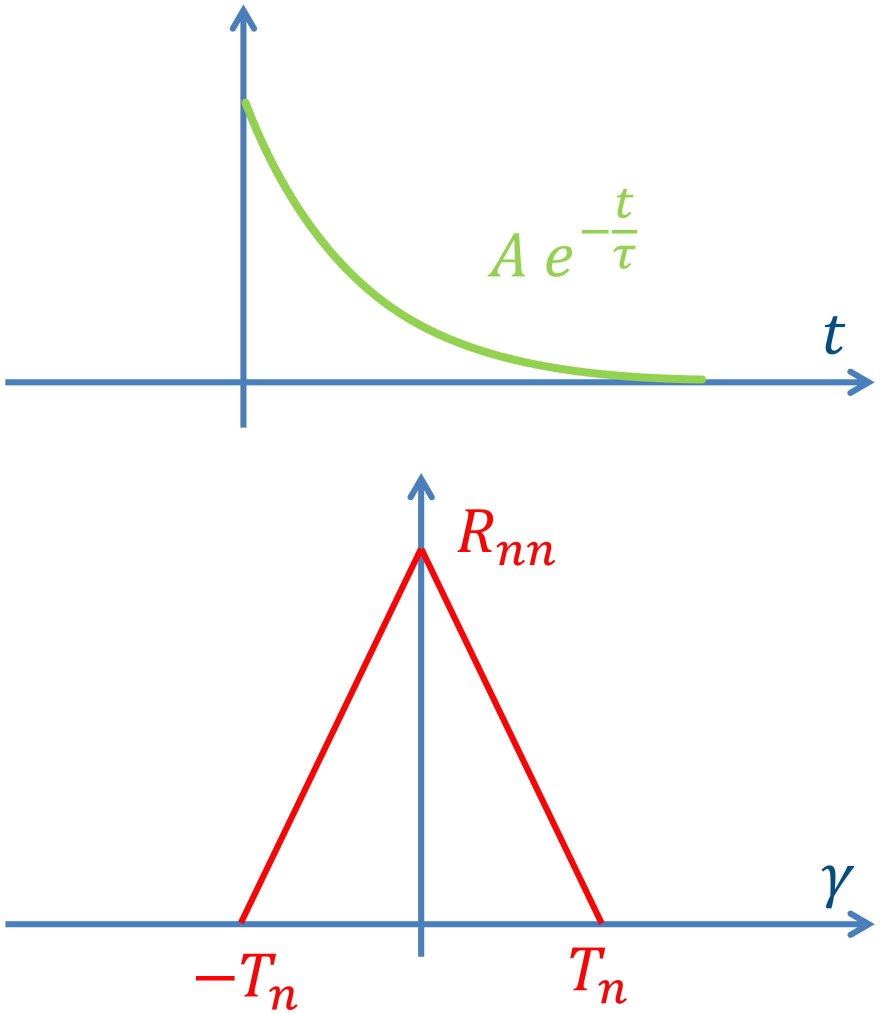
Improvement if $t_s < T_n$

Improvement

- To improve the performance, we need to reduce the WN contribution
- WN is reduced by averaging $\Rightarrow$ we could replace each samples with the average over N samples, improving S/N by $\sqrt{N}$
- The new WF becomes



Problem: exponential pulse + noise



- $\tau = 1\ \mu s$, $T_n = 10$ ns
- $S_V = 4 \times 10^{-16}$ V²/Hz

1. A uniform average is performed. Find t_s and N and evaluate the MDS
2. A weighted average is taken. Find the optimum weights and MDS
3. Compare the result with the output of a continuous-time OF
4. Compute S/N for the filter in #3 when T_n is not $\ll \tau$

Uniform average

We take a sampling time $t_s > T_n \Rightarrow$ uncorrelated noise samples

$$V_o = \sum_{k=0}^{N-1} \frac{A}{N} e^{-\frac{kt_s}{\tau}} \approx \frac{A}{N} \frac{1}{1 - e^{-\frac{t_s}{\tau}}} \qquad \overline{V_o^2} = \frac{\overline{V_i^2}}{N}$$

$$\left(\frac{S}{N}\right)_o = \frac{A}{\sqrt{\overline{V_i^2}}} \frac{1}{\sqrt{N}\left(1 - e^{-\frac{t_s}{\tau}}\right)} \approx \frac{A}{\sqrt{\overline{V_i^2}}} \sqrt{\frac{t_s}{T} \frac{\tau}{t_s}} = \left(\frac{S}{N}\right)_i \frac{\tau}{\sqrt{T t_s}}$$

Parameter values and MDS

- To improve S/N we pick the lowest value for t_s: $t_s = T_n$
- The input noise can be written as

$$\overline{V_i^2} = S_V f_n = \frac{S_V}{2T_n} \Rightarrow \left(\frac{S}{N}\right)_o = A\tau\sqrt{\frac{2}{TS_V}}$$

- Picking $T = 5\tau$ leads to $A_{min} = 31.6\ \mu\text{V}$
  Picking $T = \tau \Rightarrow A_{min} = 22.2\ \mu\text{V}$
- A single sample has $A_{min} = 141\ \mu\text{V}$

(but we must use the correct expression for the series sum)

Weighted average

For WN, the optimum weights match the signal amplitude

$$w_k = e^{-\frac{kt_s}{\tau}} \Rightarrow V_o = A \sum_{k=0}^{N-1} e^{-\frac{2kt_s}{\tau}} \approx \frac{A}{1 - e^{-\frac{2t_s}{\tau}}} \approx A \frac{\tau}{2t_s}$$

$$\overline{V_o^2} = \overline{V_i^2} \sum_{k=0}^{N-1} w_k^2 = \frac{\overline{V_i^2}}{1 - e^{-\frac{2t_s}{\tau}}} \approx \overline{V_i^2} \frac{\tau}{2t_s}$$

$$\left(\frac{S}{N}\right)_o = \frac{A}{\sqrt{\overline{V_i^2}}} \sqrt{\frac{\tau}{2t_s}} \Rightarrow A_{min} = \sqrt{\frac{S_V}{\tau}} = 20\ \mu\text{V}$$

Continuous-time OF

- If we approximate the noise as white, we can write

$$\left(\frac{S}{N}\right)_o = \frac{A}{\sqrt{S_V/2}} \sqrt{\int_0^{\infty} e^{-\frac{2t}{\tau}} dt} = A\sqrt{\frac{\tau}{S_V}} \Rightarrow A_{min} = \sqrt{\frac{S_V}{\tau}}$$

- This is no surprise: if the sampling time is small enough, DT and CT filters return the same results

Non-white noise

- The WF of the filter is

$$w(t,\gamma) = Ge^{-\frac{\gamma}{\tau}}u(\gamma) \Rightarrow k_{w_{tt}} = \frac{G^2\tau}{2}e^{-\frac{|\gamma|}{\tau}}$$

- Output signal:

$$V_o = AG\int_0^{\infty} e^{-2\gamma/\tau}d\gamma = AG\frac{\tau}{2}$$

- Output noise:

$$\overline{V_o^2} = \overline{V_i^2}G^2\tau\int_0^{T_n}\left(1-\frac{\gamma}{T_n}\right)e^{-\frac{\gamma}{\tau}}d\gamma = \overline{V_i^2}G^2\tau^2\int_0^{\frac{T_n}{\tau}}\left(1-\frac{\tau}{T_n}x\right)e^{-x}dx$$

S/N

- The output noise and S/N result in

$$\overline{V_o^2} = \overline{V_i^2} G^2 \tau^2 \left(1 - \frac{\tau}{T_n} + \frac{\tau}{T_n} e^{-\frac{T_n}{\tau}}\right)$$

$$\left(\frac{S}{N}\right)_o = \frac{A}{2\sqrt{\overline{V_i^2}}} \frac{1}{\sqrt{1 - \frac{\tau}{T_n} + \frac{\tau}{T_n} e^{-\frac{T_n}{\tau}}}} = \frac{A\sqrt{T_n}}{\sqrt{2S_V}\sqrt{1 - \frac{\tau}{T_n} + \frac{\tau}{T_n} e^{-\frac{T_n}{\tau}}}} < \left(\frac{S}{N}\right)_{opt}$$

- For $T_n \rightarrow 0$ we recover the optimum value