Electronics – 96032

POLITECNICO DI MILANO

Discrete-time Filters

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Disclaimer

Slides are supplementary material and are NOT a replacement for textbooks and/or lecture notes

Purpose of the lesson

- Most modern-day filters are realized in a digital way
- We review here a discrete-time implementation of the filters already discussed:

		Noise	
		HF (White)	LF (flicker)
Signal	LF (constant)	this lesson	next lessons
	HF (pulse)	this lesson	next lessons

Outline

- Uniform average
- Non-uniform average

Working principle

- A number N of samples of the input signal and noise are acquired with sampling time t_s
- A suitable weighted average is then performed on the data to yield the output

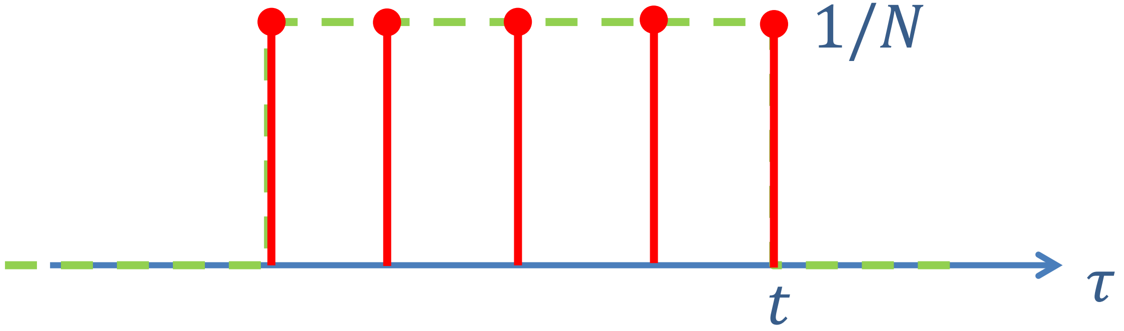
$$y = \sum_{k=1}^{N} w_k (x_k + n_k)$$

Uniform average

- In a uniform average $w_k = 1/N$

$$y = \frac{1}{N}\sum_{k=1}^{N}(x_k + n_k)$$

- We are basically building a discrete-time equivalent of the gated-integrator:



Uniform average

Constant signal $x_k = A$ and non-correlated stationary noise samples:

i.e., $T_n < t_s$

$$\bar{y} = \overline{\frac{1}{N}\sum_{k=1}^{N}(A + n_k)} = A \qquad \overline{n_y^2} = \frac{1}{N^2}\sum_{k=1}^{N}\overline{n_k^2} = \frac{\overline{n_x^2}}{N}$$

$$\left(\frac{S}{N}\right)_y = \sqrt{N}\left(\frac{S}{N}\right)_x$$

S/N improves with the square root of the number of samples

Comparison with GI

We set $T_G = T_M = N t_S$ (total measurement time)

$$\left(\frac{S}{N}\right)_{GI} = \left(\frac{S}{N}\right)_x \sqrt{\frac{T_M}{T_n}}$$

$$\left(\frac{S}{N}\right)_{AV} = \left(\frac{S}{N}\right)_x \sqrt{N} = \left(\frac{S}{N}\right)_x \sqrt{\frac{T_M}{t_S}} < \left(\frac{S}{N}\right)_{GI}$$

$$(T_n < t_S)$$

Ideal sampling WF

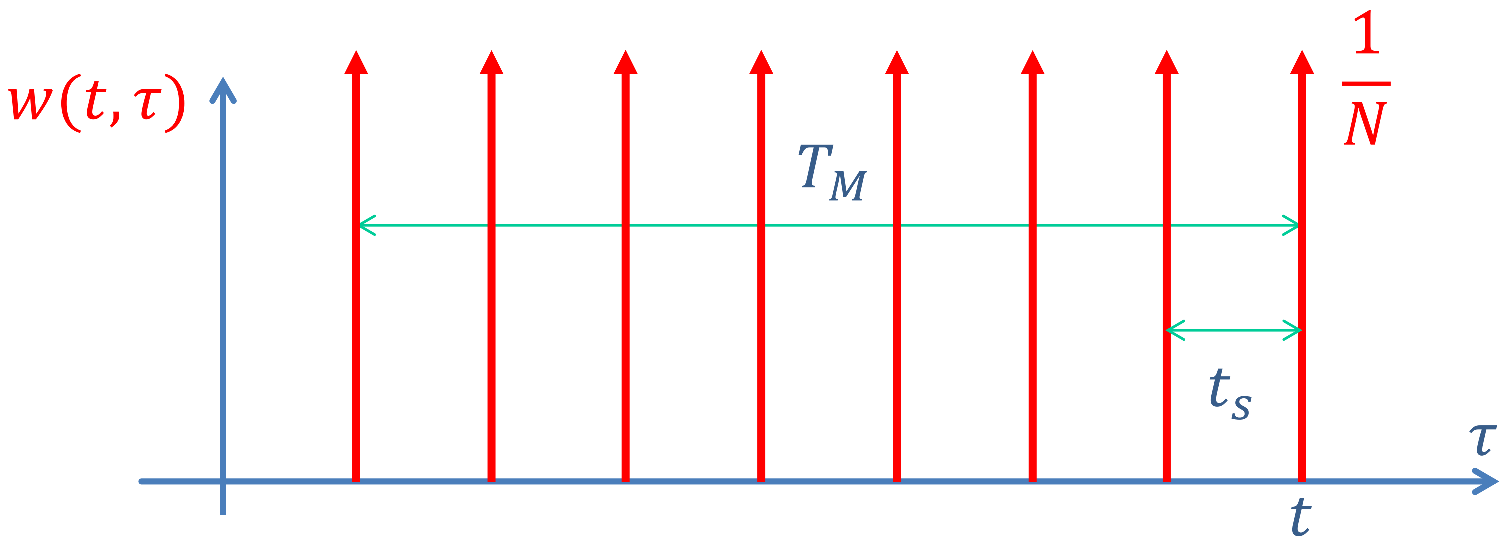
- We want to put discrete-time filters into the developed framework
- For ideal sampling at time t_s we have

$$y(t) = x(t_s),$$

which can be written as

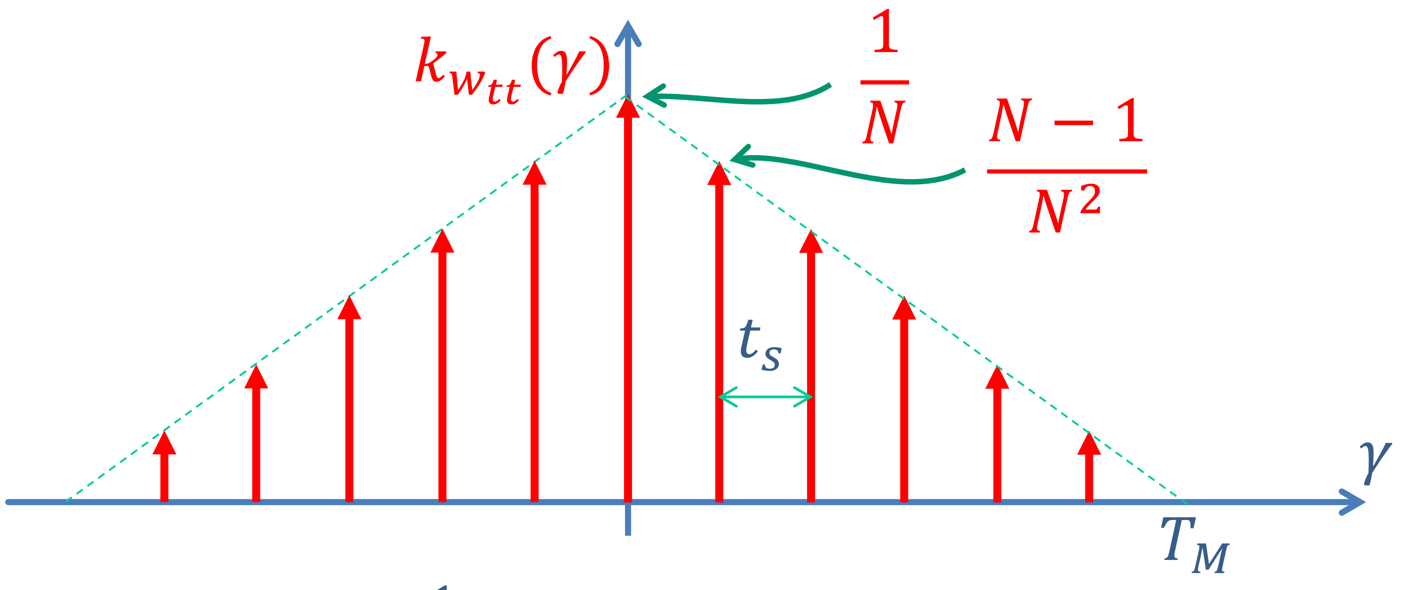
$$y(t) = \int x(\tau)\delta(t_s - \tau)d\tau \Rightarrow w(t,\tau) = \delta(t_s - \tau)$$

Weighting function – time domain



$$w(t,\tau)=\frac{1}{N}\sum_{k=0}^{N-1}\delta(\tau-(t-kt_s)) \;=\frac{1}{N}\mathrm{rect}(T_M)\sum_k\delta(\tau-kt_s)$$

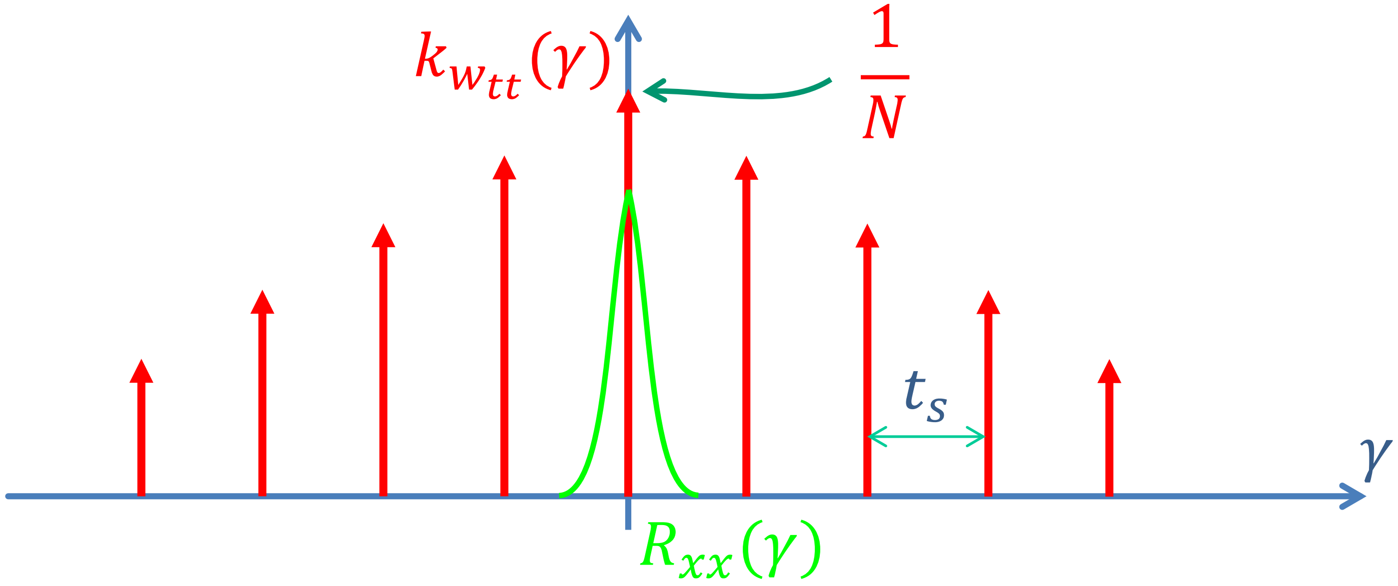
WF autocorrelation



$$k_{w_{tt}}(\gamma) = \frac{1}{N}\mathrm{tri}(T_M)\sum_k \delta(\gamma - kt_s)$$

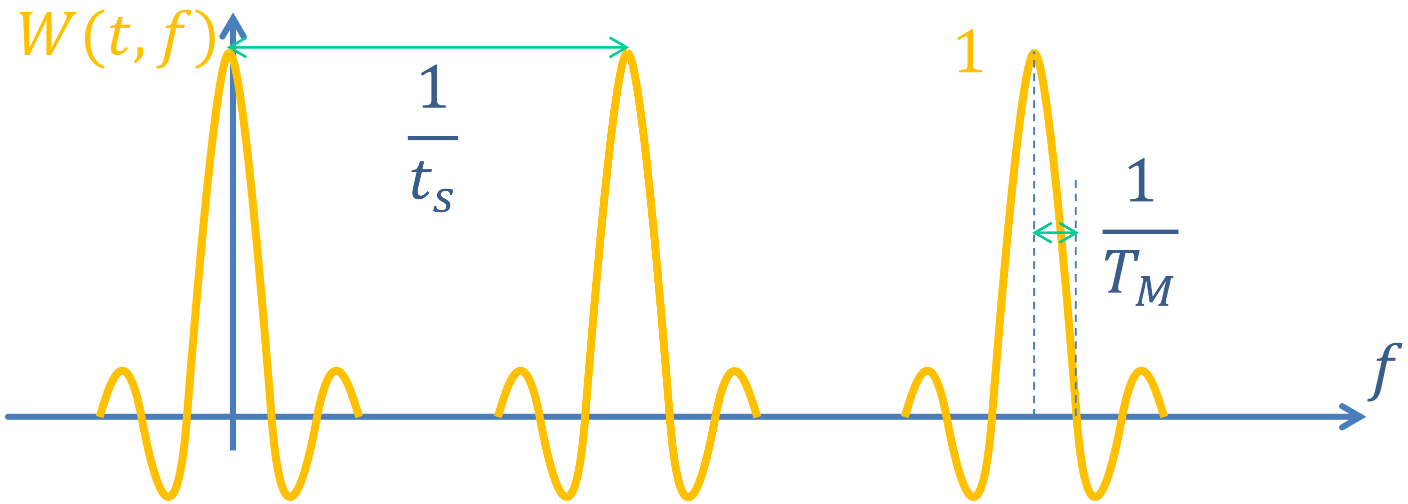
Unity amplitude

Output noise



$$\overline{n_y^2} = \int R_{xx}(\gamma) k_{w_{tt}}(\gamma) d\gamma = \frac{\overline{n_x^2}}{N}$$

Weighting function – frequency domain

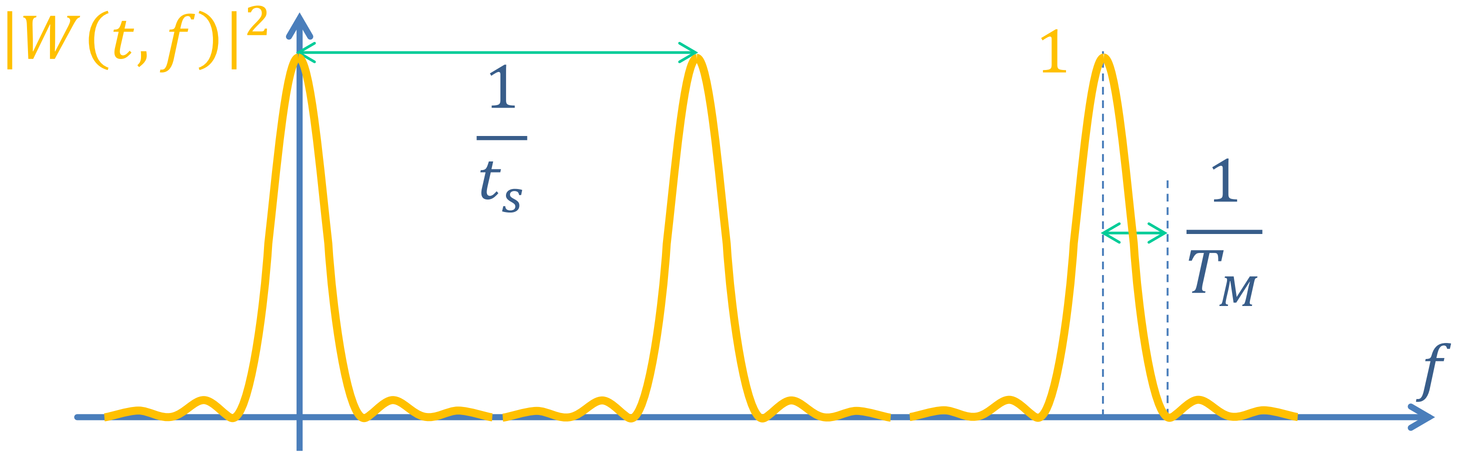


$$W(t,f) = \frac{T_M}{N}\operatorname{sinc}(\pi f T_M) * \frac{1}{t_s}\sum_k \delta\left(f - \frac{k}{t_s}\right)$$

$$= \sum_k \operatorname{sinc}\left[\pi T_M\left(f - \frac{k}{t_s}\right)\right]$$

(we neglect the phase term associated to the rectangle not being centered around $\tau = 0$)

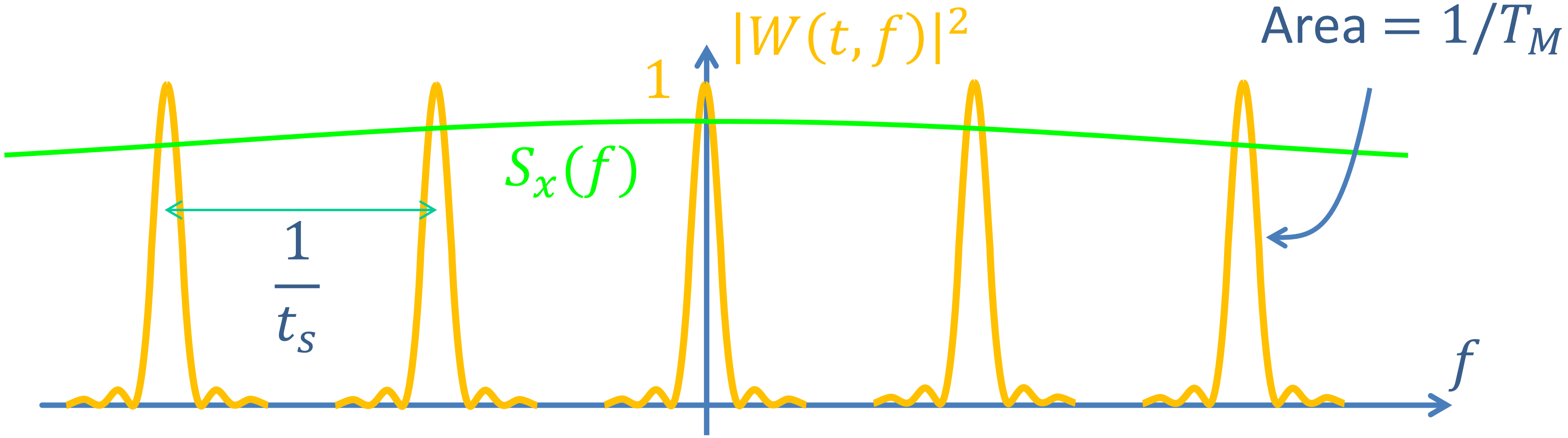
WF autocorrelation



$$|W(t,f)|^2 = \frac{T_M}{N}\operatorname{sinc}^2(\pi f T_M) * \frac{1}{t_s}\sum_k \delta\left(f - \frac{k}{t_s}\right)$$

$$= \sum_k \operatorname{sinc}^2\left[\pi T_M\left(f - \frac{k}{t_s}\right)\right]$$

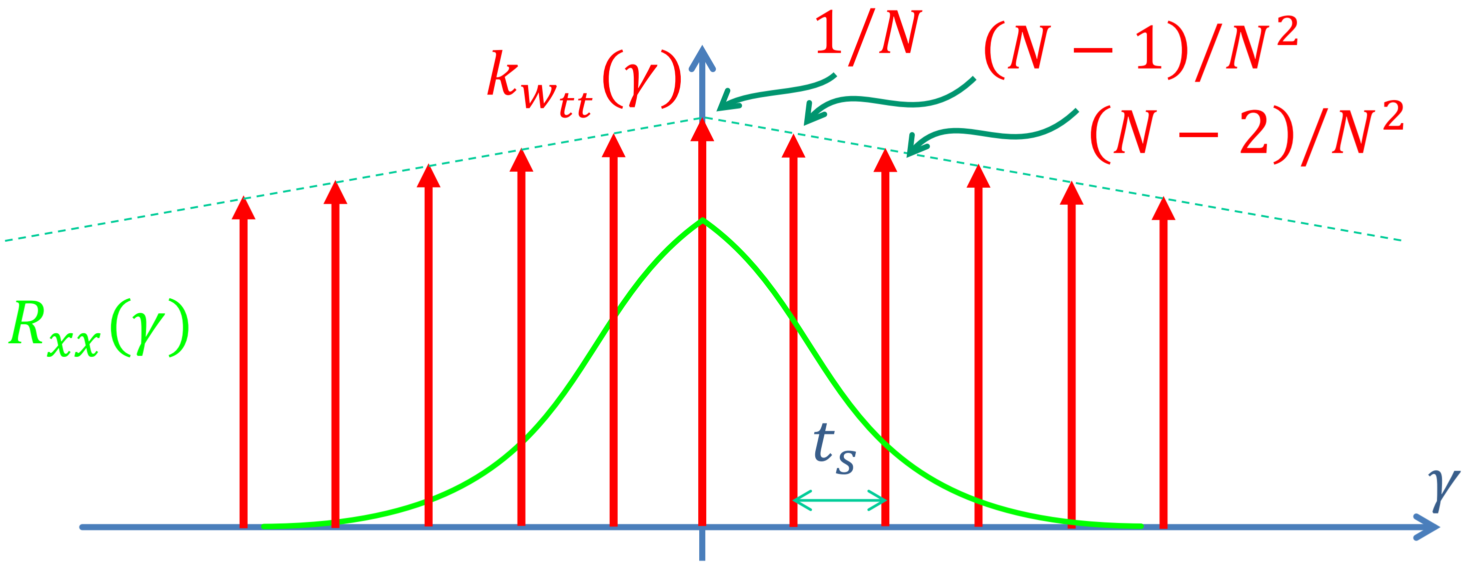
Output noise



$$\overline{n_y^2} = \int S_x(f)|W(t,f)|^2 df = \frac{1}{T_M}\sum_k S_x\left(\frac{k}{t_s}\right) = \frac{1}{N}\sum_k \frac{1}{t_s} S_x\left(\frac{k}{t_s}\right)$$

$$= \overline{n_x^2}/N$$

Correlated input noise – time domain



We consider a noise that is correlated over a few samples, where the δ-function amplitude is about $1/N$:

$$\overline{n_y^2} = \int R_{xx}(\gamma) k_{w_{tt}}(\gamma) d\gamma \approx \frac{1}{N} \sum_k R_{xx}(k t_s)$$

Correlated input noise – frequency domain

- In the frequency domain we get

$$\overline{n_y^2} = \frac{1}{N}\sum_k \frac{1}{t_s} S_x\left(\frac{k}{t_s}\right),$$

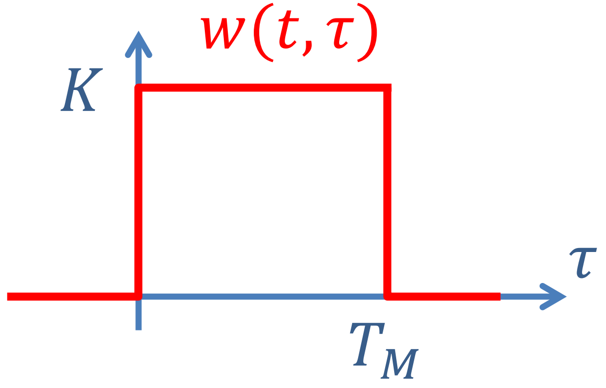
considering the sinc² functions as δ-functions with respect to S_x (same as saying that the correlation time is much shorter than T_M)

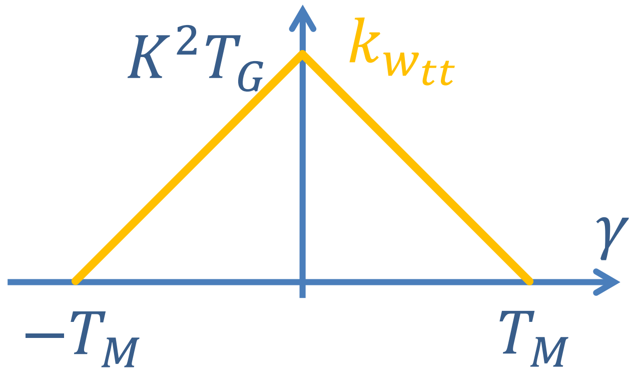
- The two expressions are equivalent via Parseval theorem

Comparison against GI

- DT filter has unity signal gain $\Rightarrow$ for GI to have the same gain, we set $T_G = T_M$ and $K = 1/T_G$
- Output noise (correlation time much shorter than T_M):

$$\overline{n_{GI}^2} = \int R_{xx}(\gamma) k_{w_{tt}}(\gamma) d\gamma \approx \frac{1}{T_M} \int R_{xx}(\gamma) d\gamma = \frac{1}{T_M} S_x(0)$$





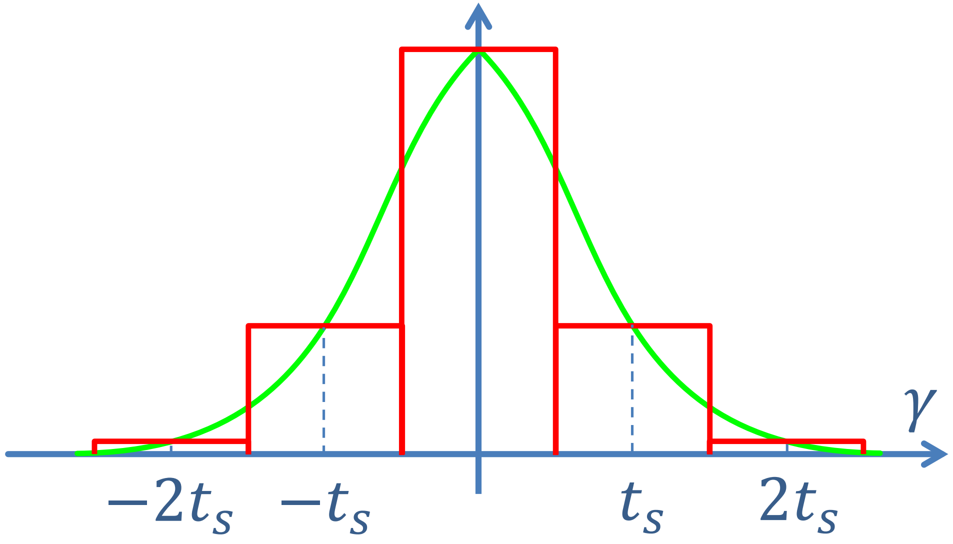
Comparison against GI

- Time domain:

$$\frac{\overline{n_{AV}^2}}{\overline{n_{GI}^2}} = \frac{\frac{1}{N}\sum_k R_{xx}(kt_s)}{\frac{1}{T_M}\int R_{xx}(\gamma)d\gamma} = \frac{t_s \sum_k R_{xx}(kt_s)}{\int R_{xx}(\gamma)d\gamma} > 1$$

- Frequency domain:

$$\frac{\overline{n_{AV}^2}}{\overline{n_{GI}^2}} = \frac{\sum S_x\left(\frac{k}{t_s}\right)}{S_x(0)} > 1$$



Outline

- Uniform average
- Non-uniform average

Non-uniform average

- Is used to mimic the behavior of any CT filter (or those unfeasible in CT)
- We consider once again constant signal and non-correlated stationary noise samples ($T_n < t_s$)

$$\bar{y} = \overline{\sum_{k=1}^{N} w_k (x_k + n_k)} = \sum_{k=1}^{N} w_k x_k = A \sum_{k=1}^{N} w_k$$

$$\overline{n_y^2} = \sum_{k=1}^{N} w_k^2 \overline{n_k^2} = \overline{n_x^2} \sum_{k=1}^{N} w_k^2$$

Improvement in S/N

$$\left(\frac{S}{N}\right)_y = \left(\frac{S}{N}\right)_x \frac{\sum_{k=1}^{N} w_k}{\sqrt{\sum_{k=1}^{N} w_k^2}}$$

For constant weights, the improvement with $\sqrt{N}$ is recovered

Power-law weighting

We set $w_k = \alpha^k$:

$$\bar{y} = A \sum_{k=0}^{N-1} \alpha^k \approx A \frac{1}{1-\alpha} \qquad \overline{n_y^2} = \overline{n_x^2} \sum_{k=0}^{N-1} \alpha^{2k} = \overline{n_x^2} \frac{1}{1-\alpha^2}$$

$$\left(\frac{S}{N}\right)_y = \left(\frac{S}{N}\right)_x \sqrt{\frac{1+\alpha}{1-\alpha}}$$

N_{eq}

Diverging as $\alpha \to 1$??

Equivalent number of samples

- If α^N does not go to zero, the partial sum ought to be considered and the result becomes:

$$N_{eq} = \frac{1+\alpha}{1-\alpha}\frac{1-\alpha^N}{1+\alpha^N}$$

- $N_{eq} \leq N$ and $N_{eq} \to N$ for $\alpha \to 1 \Rightarrow$ the best filtering option in this case is uniform

Example: discrete-time LPF

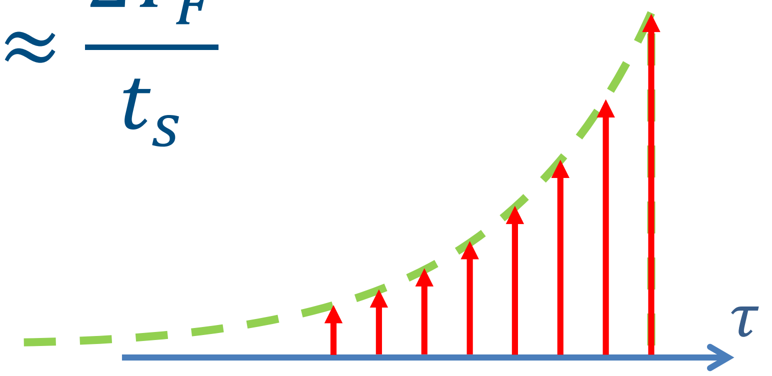
- The weights are

$$w_k = e^{-kt_s/T_F} = \left(e^{-t_s/T_F}\right)^k \Rightarrow \alpha = e^{-t_s/T_F}$$

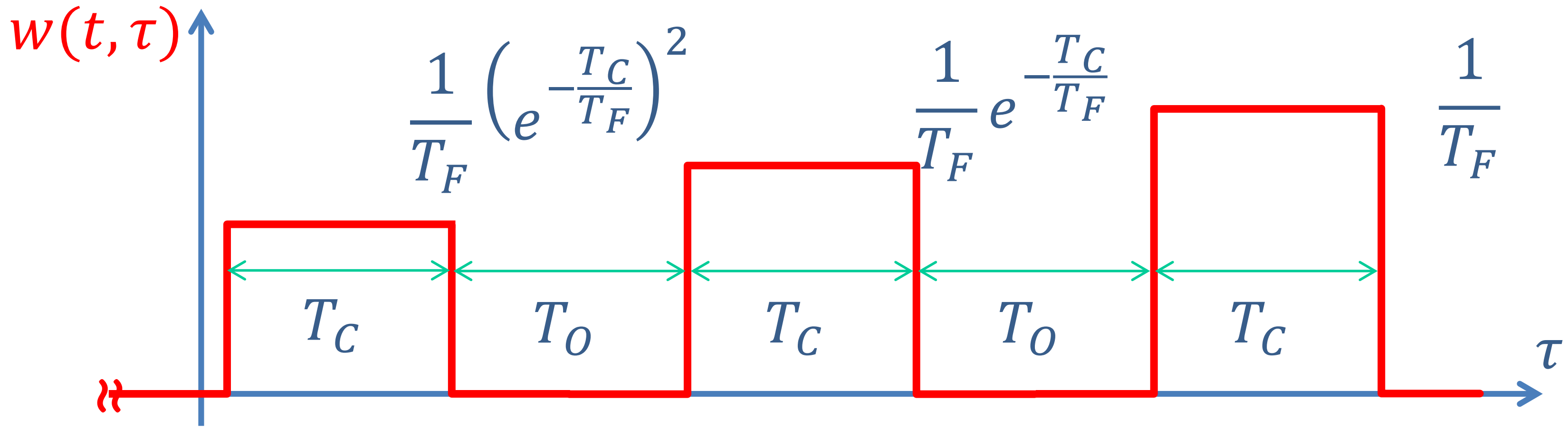
- The improvement in S/N goes with $\sqrt{N_{eq}}$, where

$$N_{eq} = \frac{1+\alpha}{1-\alpha} = \frac{1+e^{-t_s/T_F}}{1-e^{-t_s/T_F}} \approx \frac{2T_F}{t_s}$$

- In the continuous time we had $\frac{2T_F}{T_n} \geq N_{eq}$



BA as an averaging filter



- BA can be seen as cascade of two filters:
  - Single-pulse BA with gain $1/T_F$
  - Power-law average of samples with $\alpha = e^{-T_C/T_F}$

Equivalent number of samples

- We recover the BA results:

$$N_{eq} = \frac{1+\alpha}{1-\alpha} = \frac{1+e^{-T_C/T_F}}{1-e^{-T_C/T_F}} \approx \frac{2T_F}{T_C}$$

- Analogously, for the ratemeter

$$\alpha = e^{-(T_C+T_O)/T_F} \Rightarrow N_{eq} = \frac{2T_F}{T_C+T_O}$$

(both approximations valid for $T_C + T_O \ll T_F$)