POLITECNICO DI MILANO

Electronics – 96032

Optimum Filtering

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Disclaimer

Slides are supplementary material and are NOT a replacement for textbooks and/or lecture notes

Purpose of the lesson

Well, we are still here, but we consider the general problem and somehow start to address the case of non-white noise

		Noise	
		HF (White)	LF (flicker)
Signal	LF (constant)	this lesson	next lessons
	HF (pulse)	this lesson	next lessons

Outline

- Optimum filter for white noise
- The non-white noise case
- Non-stationary white noise

The problem

- In many physics and engineering applications the signal has a fixed and known shape, and depends upon several parameters
- The purpose of filtering is then to make a precise measurement of the signal parameters (amplitude, arrival time,…)
- We want to determine the optimum filter, which gives the best possible output S/N

DT filter, WN

- We start with a DT filter and white input noise
- The sampled input signal is $Ax_k = Ax(kt_s)$ (amplitude is the parameter of interest)

$$\bar{y} = \overline{\sum_{k=1}^{N} w_k(Ax_k + n_k)} = A\sum_{k=1}^{N} w_k x_k$$

$$\overline{n_y^2} = \sum_{k=1}^{N} w_k^2 \overline{n_k^2} = \overline{n_x^2} \sum_{k=1}^{N} w_k^2$$

Output S/N

$$\left(\frac{S}{N}\right)_y = \left(\frac{S}{N}\right)_x \frac{\sum_{k=1}^{N} w_k x_k}{\sqrt{\sum_{k=1}^{N} w_k^2}}$$

We must determine the optimum choice for the weights w_k, maximizing $\left(\frac{S}{N}\right)_x$:

$$\frac{\partial}{\partial w_n}\left(\frac{S}{N}\right)_x = 0 \quad \forall\, n = 1, \ldots, N$$

Optimum weights

$$\frac{x_n\sqrt{\sum_{k=1}^{N} w_k^2} - \frac{w_n}{\sqrt{\sum_{k=1}^{N} w_k^2}}\sum_{k=1}^{N} w_k x_k}{\sum_{k=1}^{N} w_k^2} = 0$$

$$w_n = x_n \frac{\sum_{k=1}^{N} w_k^2}{\sum_{k=1}^{N} w_k x_k}$$

The optimum weights are proportional to the signal amplitude

CT filter, WN

$$y = A \int x(\tau) w(t,\tau) d\tau$$

$$\overline{n_y^2} = \lambda k_{w_{tt}}(0) = \lambda \int w^2(t,\tau) d\tau$$

$$\left(\frac{S}{N}\right)_y^2 = \frac{A^2}{\lambda} \frac{|\int x(\tau) w(t,\tau) d\tau|^2}{\int w^2(t,\tau) d\tau}$$

Optimum weighting function

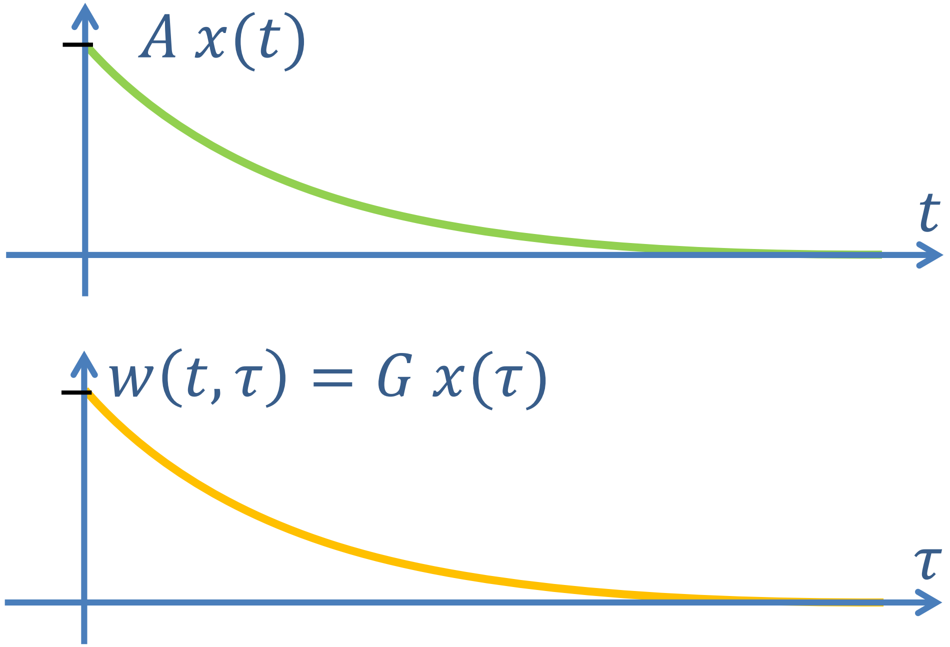
- Schwarz inequality in $\boldsymbol{L}^2$:

$$|\langle x, w\rangle| \leq \|x\| \cdot \|w\| \Rightarrow$$

$$\left|\int x(\tau) w(t,\tau) d\tau\right|^2 \leq \int |x(\tau)|^2 d\tau \int |w(t,\tau)|^2 d\tau$$

- The maximum is achieved when $w(t,\tau) \propto x(\tau)$ (weighting function proportional to signal amplitude)

Matched filter



The filter weights more the regions with higher signal and discards those where it is negligible (it «matches» the signal)

Optimum S/N

$$w(t,\tau) = G\,x(\tau) \Rightarrow y = AG\int x^2(\tau)d\tau = AGk_{xx}(0)$$

$$\overline{n_y^2} = \lambda k_{w_{tt}}(0) = \lambda G^2\int x^2(\tau)d\tau$$

$$\left(\frac{S}{N}\right)_y = \frac{A}{\sqrt{\lambda}}\sqrt{\int x^2(t)dt} = \sqrt{\frac{E}{\lambda}}$$

Frequency domain

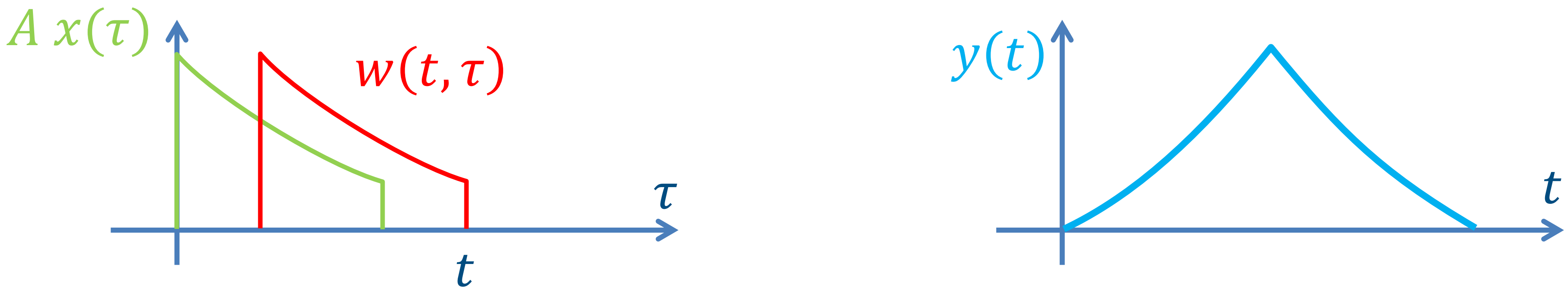
Optimum filtering can also be derived in the frequency domain

$$\left(\frac{S}{N}\right)_y^2 = \frac{A^2}{\lambda}\frac{\left|\int X(f)W^*(t,f)df\right|^2}{\int |W(t,f)|^2 df}$$

$$\leq \frac{A^2}{\lambda}\frac{\int |X(f)|^2 df \int |W(t,f)|^2 df}{\int |W(t,f)|^2 df} = \frac{A^2}{\lambda}\int |X(f)|^2 df$$

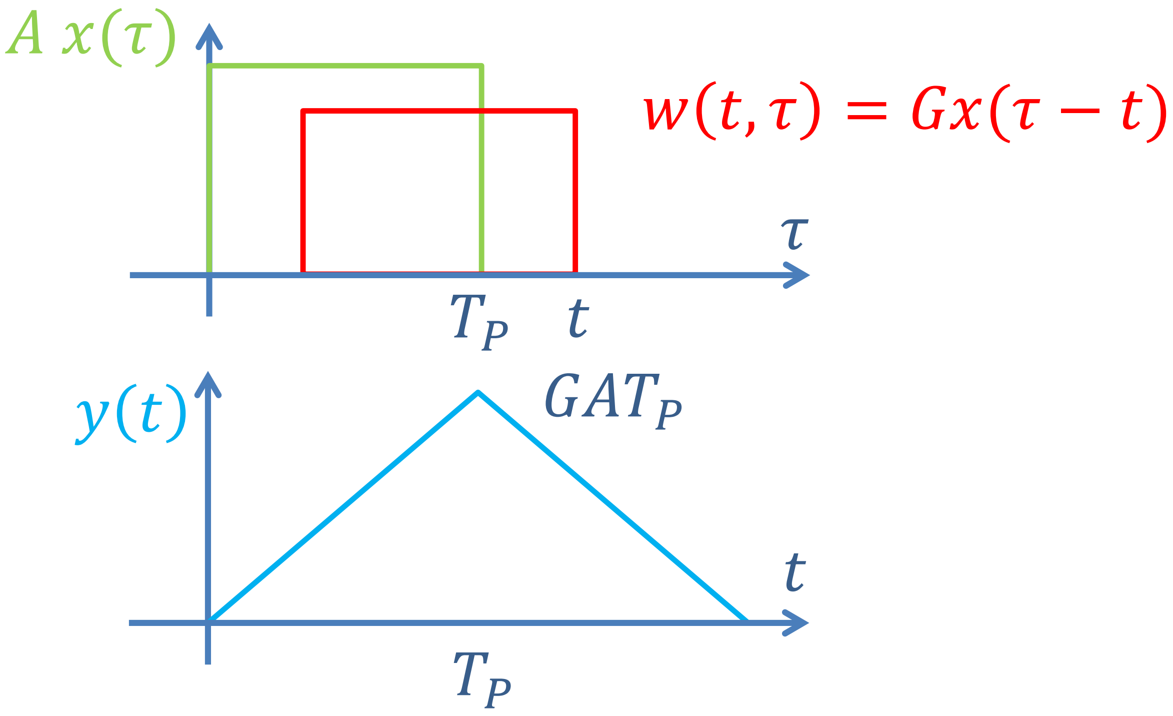
(equality holds only if $W(t,f) \propto X(f)$)

Input/output signals



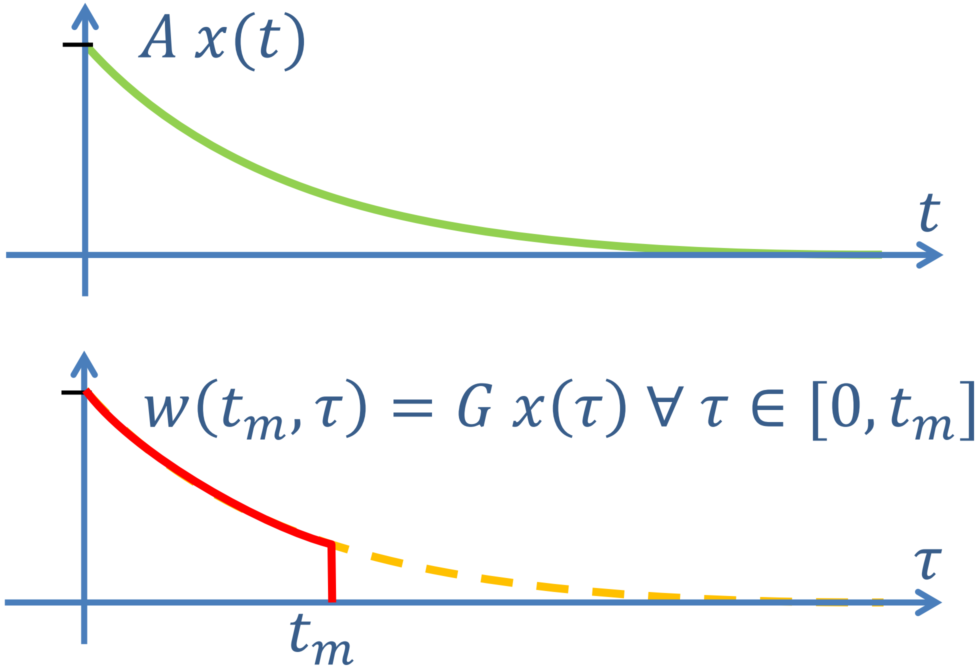
For time-limited signals, the output is the time correlation of the input signal ⇒ the matched filter is sometimes called correlator

Finite pulse time



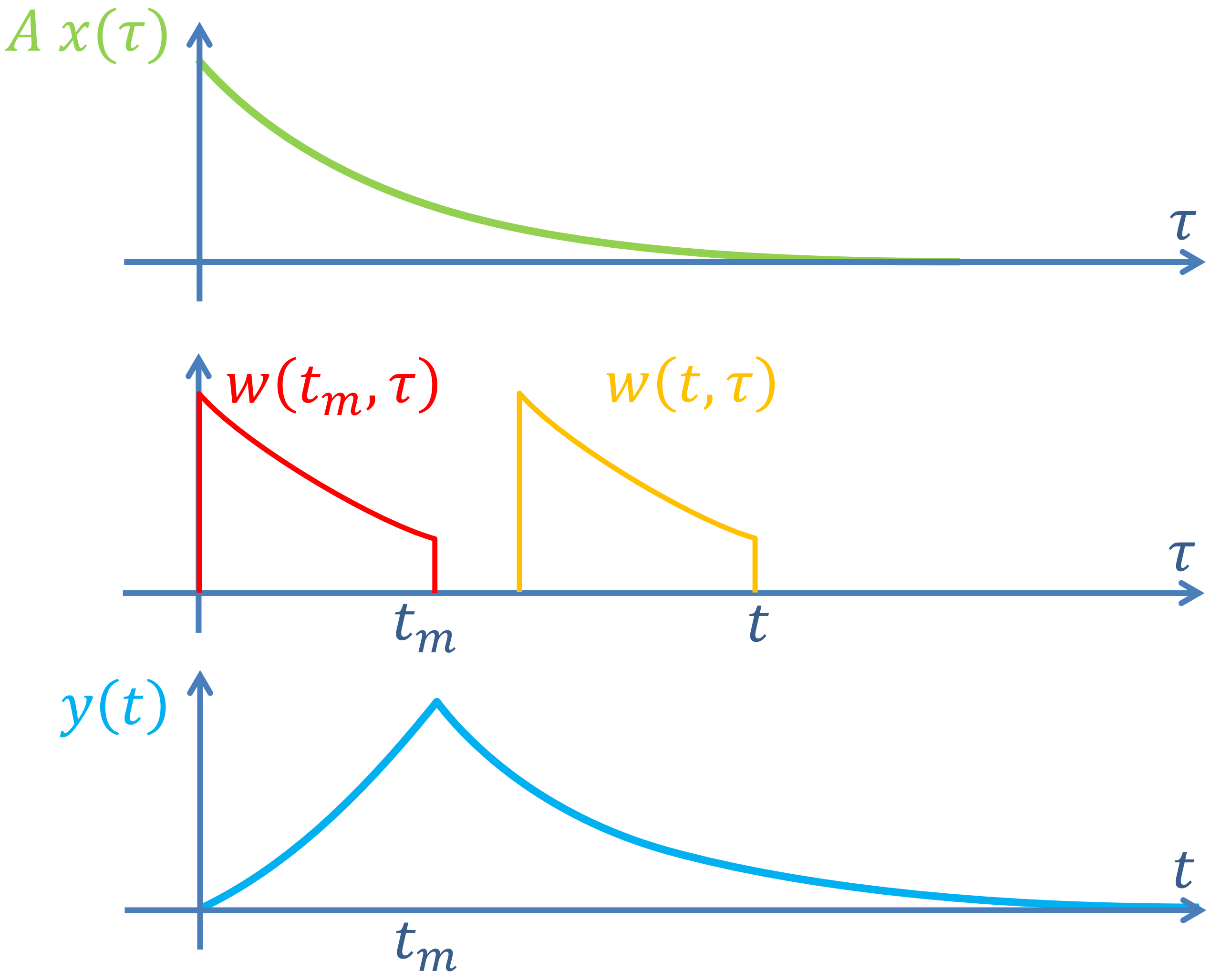
A time equal to T_P is (obviously) needed to process the pulse

Finite measurement time

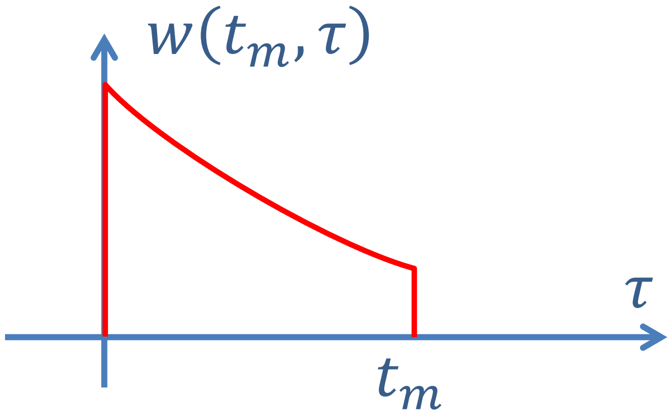


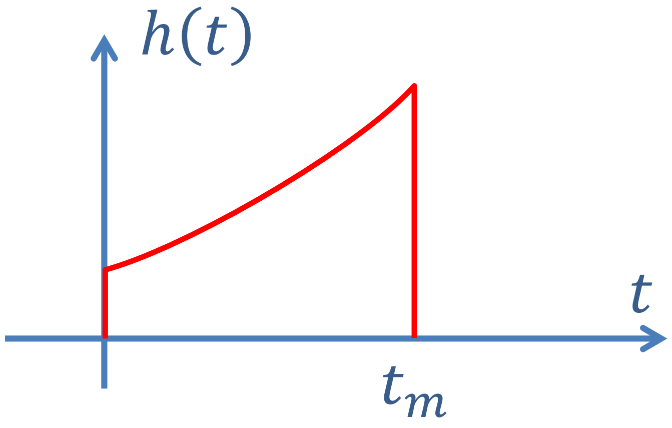
A finite readout time is required in real filters $\Rightarrow$ the weighting function matches the signal only in the interval of interest

Input/output signals



δ-function response





- If the filter is LTI, $w(t,\tau) = h(t-\tau)$ where $h(t)$ is the δ–function response $\Rightarrow$ often difficult or impossible to build
- The filter is usually time-variant or digital
- Approximations are often used to simplify the filter design with acceptable performance degradation

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Non-white noise

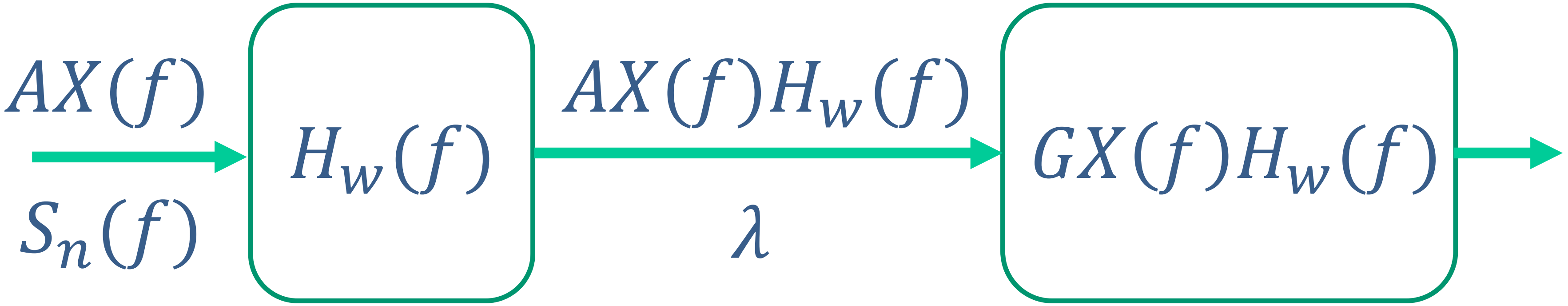
- Every noise $S_n(f)$ can be transformed into white by a linear (whitening) filter $H_w(f)$:

$$S_n(f)|H_w(f)|^2 = \lambda$$

$$|H_w(f)| = \sqrt{\frac{\lambda}{S_n(f)}}$$

- A matched filter can now be used on the output signal $X_w(f) = H_w(f)AX(f)$

Optimum filter



$$\left(\frac{S}{N}\right)_y^2 = \frac{A^2}{\lambda}\int |X(f)|^2 |H_w(f)|^2 df = A^2 \int \frac{|X(f)|^2}{S_n(f)} df$$

The optimum filter WF in the frequency domain is

$$W(f) = GX(f)|H_w(f)|^2 = G' \frac{X(f)}{S_n(f)}$$

Analytical derivation

$$\left(\frac{S}{N}\right)_y^2 = A^2 \frac{\left|\int X(f)W^*(t,f)df\right|^2}{\int |W(t,f)|^2 S_n(f)df}$$

(multiply and divide by $\sqrt{S_n(f)}$)

$$\leq A^2 \frac{\int \frac{|X(f)|^2}{S_n(f)}df \int S_n(f)|W^*(t,f)|^2 df}{\int |W(t,f)|^2 S_n(f)df} = A^2 \int \frac{|X(f)|^2}{S_n(f)}df$$

Considerations

- The WF in the frequency domain is again intuitive: the weights are given by the elementary S/N contributions $X(f)/S_n(f)$
- Splitting the filter into a whitening + matched part is only a matter of interpretation; the filter output only depends on the global weighting function
- If the noise is Gaussian, the filter is optimum even with respect to non-linear alternatives

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Non-stationary white noise

- Noise autocorrelation and PSD are

$$R_{xx}(t_1, t_2) = \lambda(t_1)\delta(t_2 - t_1) \Leftrightarrow S_n(t, f) = \lambda(t)$$

- The output noise is given by

$$\overline{n_y^2(t)} = \iint R_{xx}(\alpha, \beta) w(t, \alpha) w(t, \beta) d\alpha d\beta = \int \lambda(\alpha) w^2(t, \alpha) d\alpha$$

- The signal output is $y = A \int x(\tau) w(t, \tau) d\tau \Rightarrow$ the expression is analogous to the previous one and the optimum filter is

$$w(t, \tau) = G \frac{x(\tau)}{\lambda(\tau)}$$

Example: the shot noise case

- Consider a current $I(t)$ affected by shot noise, having bilateral $S_n = qI(t)$
- The optimum filter becomes now

$$w(t,\tau) = G\frac{I(t)}{qI(t)} = \text{const}$$

i.e., a gated integrator!

- If other noise sources are present (thermal noise,…), $w(t,\tau)$ is modified accordingly

References

In http://home.deib.polimi.it/cova/elet/lezioni/SSN06b_Filters-OPF2.pdf

you will find several examples of optimum filter design