POLITECNICO DI MILANO

Electronics – 96032

Signals and Noise

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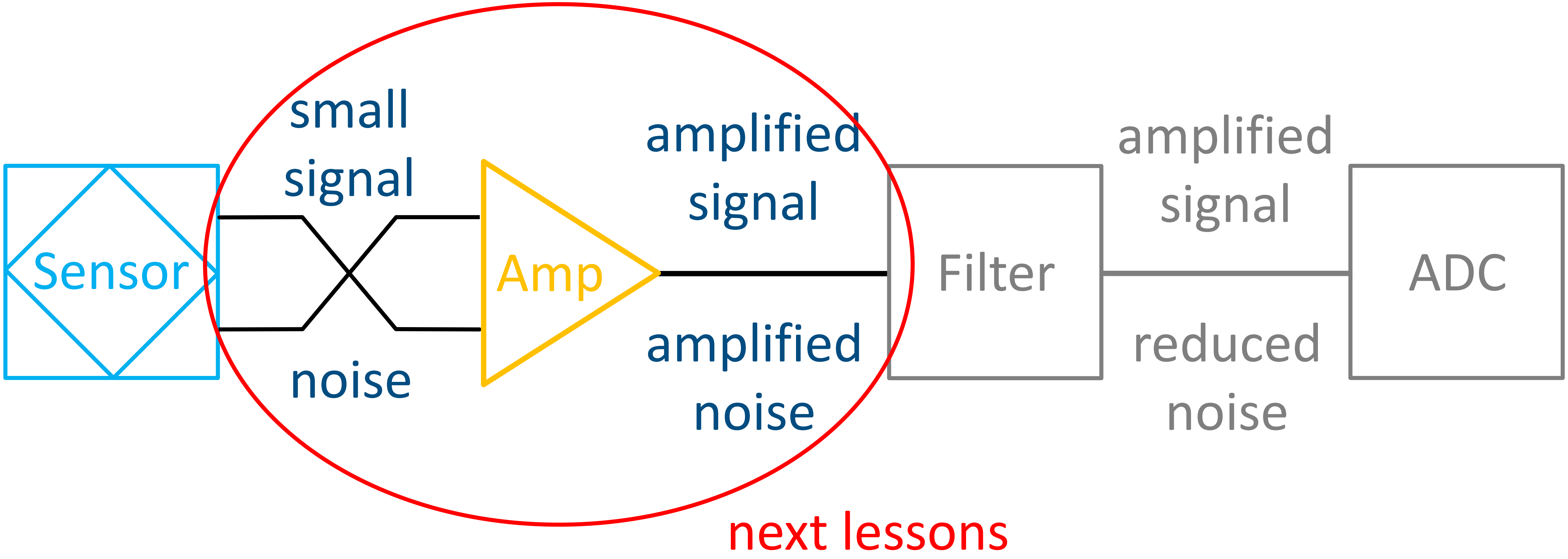
Disclaimer

Slides are supplementary material and are NOT a replacement for textbooks and/or lecture notes

Acquisition chain



Purpose of the lesson

- Up to now we have seen Amplifiers and Sensors and we know how to deal with the problems from the viewpoint of the signal
- In reality, noise is always present and can affect the precision or even overshadow the signal
- In the second half of the class, we will discuss techniques for noise reduction. Before doing this, however, we need to learn:
  - how to mathematically describe noise (this lesson);
  - what are the origins of noise (next lesson);
  - how circuits respond to noise (lesson after the next);

Outline

- Signals in time and frequency domains
- Random processes
- White noise and approximations

Signals

- Physical quantities that vary with time and contain information
- Deterministic in nature, and described by their time or frequency behavior

Fourier transform

$$X(\omega) = \int_{-\infty}^{+\infty} x(t) e^{-j\omega t} dt \qquad X(f) = \int_{-\infty}^{+\infty} x(t) e^{-j2\pi f t} dt$$

$$x(t) = \int_{-\infty}^{+\infty} X(\omega) e^{j\omega t} \frac{d\omega}{2\pi} = \int_{-\infty}^{+\infty} X(f) e^{j2\pi f t} df$$

It is basically a Laplace transform evaluated for $s = j2\pi f$

General properties

- Linearity

$$\mathcal{F}[ax(t) + by(t)] = aX(f) + bY(f)$$

- Initial values

$$X(0) = \int x(t)dt \qquad x(0) = \int X(f)df$$

- Symmetry
  - If $x(t)$ is even, so is $X(f)$
  - If $x(t)$ is real and even, so is $X(f)$

Properties – differentiation

	Time	Frequency
Functions	$x(t)$	$X(f)$
Time differentiation	$x'(t)$	$j2\pi f\, X(f)$
Freq. differentiation	$-j2\pi t\, x(t)$	$X'(f)$

Properties – integration and shift

	Time	Frequency
Time integration	$\int_{-\infty}^{t} x(\tau)d\tau$	$\frac{X(f)}{j2\pi f} + \frac{X(0)}{2}\delta(f)$
Time shifting	$x(t+\tau)$	$e^{j2\pi f\tau}\ X(f)$
Frequency shifting	$e^{-j2\pi f_0 t}x(t)$	$X(f+f_0)$

Integration example

$$x(t) = \delta(t) \Leftrightarrow X(f) = \int \delta(t) e^{-j2\pi f t} dt = 1$$

$$y(t) = \int_{-\infty}^{t} x(\tau) d\tau = u(t) + C \Leftrightarrow \frac{X(f)}{j2\pi f} = \frac{1}{j2\pi f}$$

For the initial value theorem:

$$y(0) = \int \frac{1}{j2\pi f} df = 0,$$

which means that $\frac{1}{j2\pi f}$ is the Fourier transform of the symmetric step function, with $C = -1/2$.
The Transform of $u(t)$ becomes then

$$u(t) \Leftrightarrow \frac{1}{j2\pi f} + \frac{1}{2}\delta(f)$$

Properties – scaling

	Time	Frequency
Time scaling	$x(at)$	$\frac{1}{\lvert a\rvert} X\left(\frac{f}{a}\right)$
Time reversal	$x(-t)$	$X(-f)$ $\left(\begin{array}{c}\text{if } x(t) \text{ is real} \\ X(-f) = X^*(f)\end{array}\right)$

Properties – convolution

	Time	Frequency
Time domain	$x(t) * y(t)$	$X(f)Y(f)$
Frequency domain	$x(t)y(t)$	$X(f) * Y(f)$

where $$x(t) * y(t) = \int x(\tau)y(t-\tau)d\tau$$

Parseval theorem

$$\int x(t)y^*(t)dt = \int X(f)Y^*(f)df$$

and for $x = y$

$$\int |x(t)|^2 dt = \int |X(f)|^2 df$$

Time-bandwidth relation

- The Fourier transform of a Gaussian signal is also Gaussian:

$$\mathcal{F}\left[e^{-\pi\sigma^2 t^2}\right] = e^{-\pi f^2/\sigma^2}$$

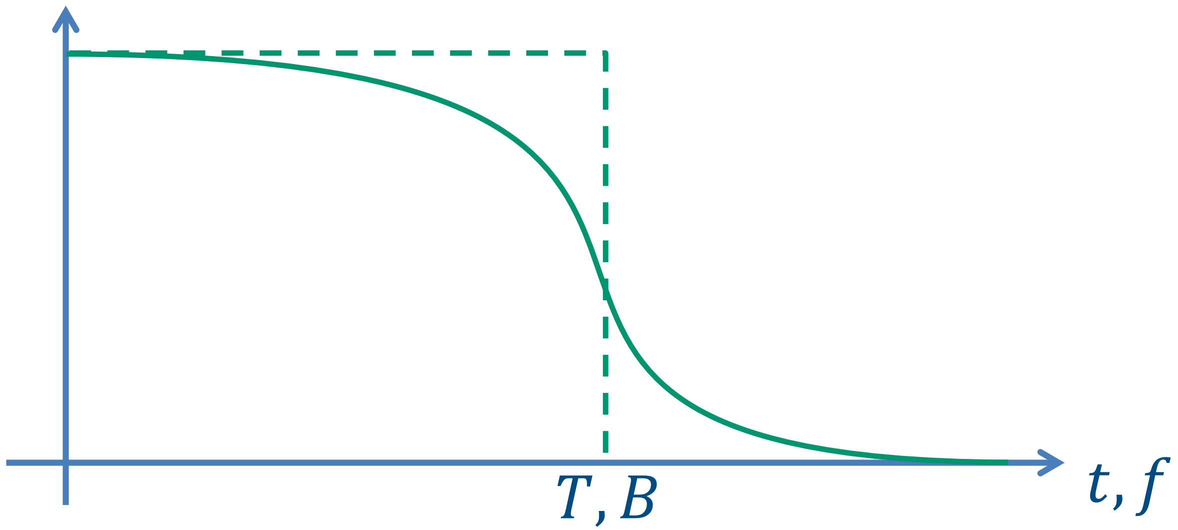
- The widths of the functions are related by an inverse relation

$$\sigma_t \sigma_f = 1$$

Extension to the general case

We consider the equivalent duration or bandwidth:

$$\int x(t)dt = x(0)T \qquad \int X(f)df = X(0)B$$



Extension to the general case

From the initial-value theorem:

$$X(0) = \int x(t)dt = x(0)T$$

$$x(0) = \int X(f)df = X(0)B_f = x(0)TB_f$$

$$\Rightarrow TB_f = 1$$

(becomes $TB_\omega = 2\pi$ in the ω domain)

Signal cross-correlation

- We consider real signals belonging to $L^2(\mathcal{R})$, called energy signals
- Cross-correlation of two energy signals is defined as

$$k_{xy}(\tau) = \int x(t)y(t+\tau)dt$$

and measures the "similarity" between the signals, as a function of their time difference τ

General properties

$$k_{xy}(\tau) = \int x(t)y(t+\tau)dt = \int x(z-\tau)y(z)dz = k_{yx}(-\tau)$$

$$|k_{xy}(\tau)| \leq \sqrt{k_{xx}(0)k_{yy}(0)}$$

$$|k_{xy}(\tau)| \leq \frac{1}{2}\left(k_{xx}(0) + k_{yy}(0)\right)$$

Autocorrelation

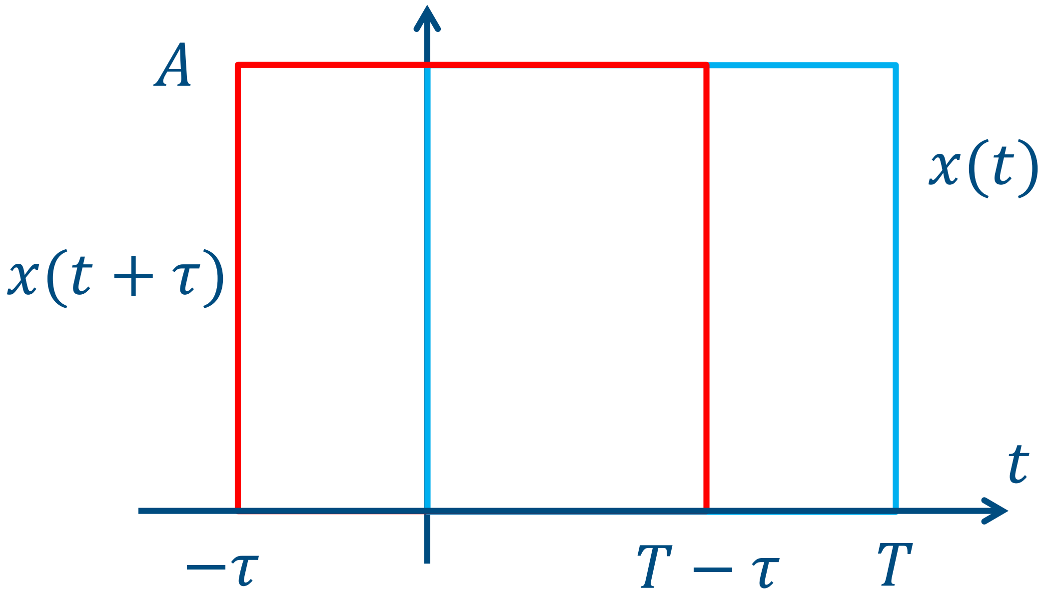
- Is the correlation of a signal with itself
- It measures the "predictability" of the signal over time

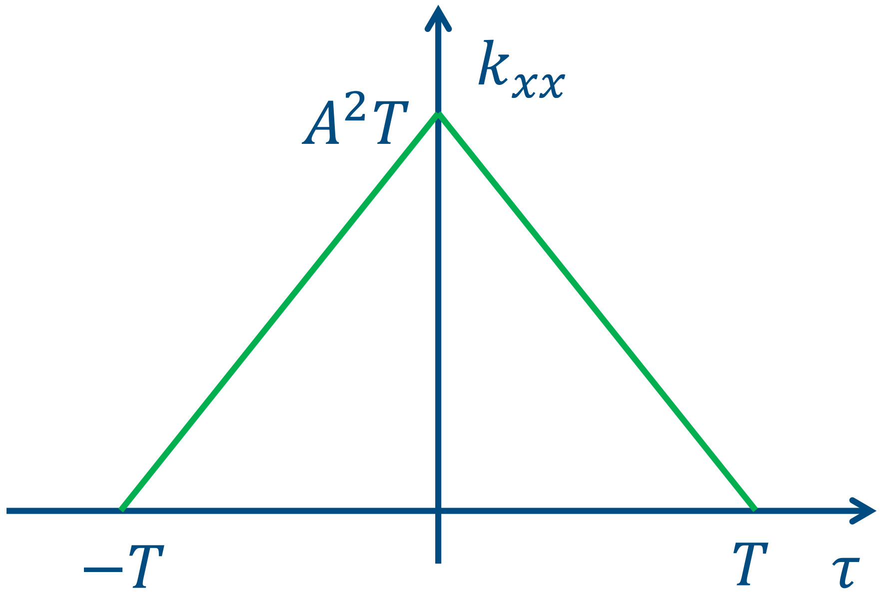
$$k_{xx}(\tau) = \int x(t)x(t+\tau)dt = k_{xx}(-\tau)$$ (Real and even)

$$|k_{xx}(\tau)| \leq k_{xx}(0) = \int x^2(t)dt = E$$

- E is called the signal energy

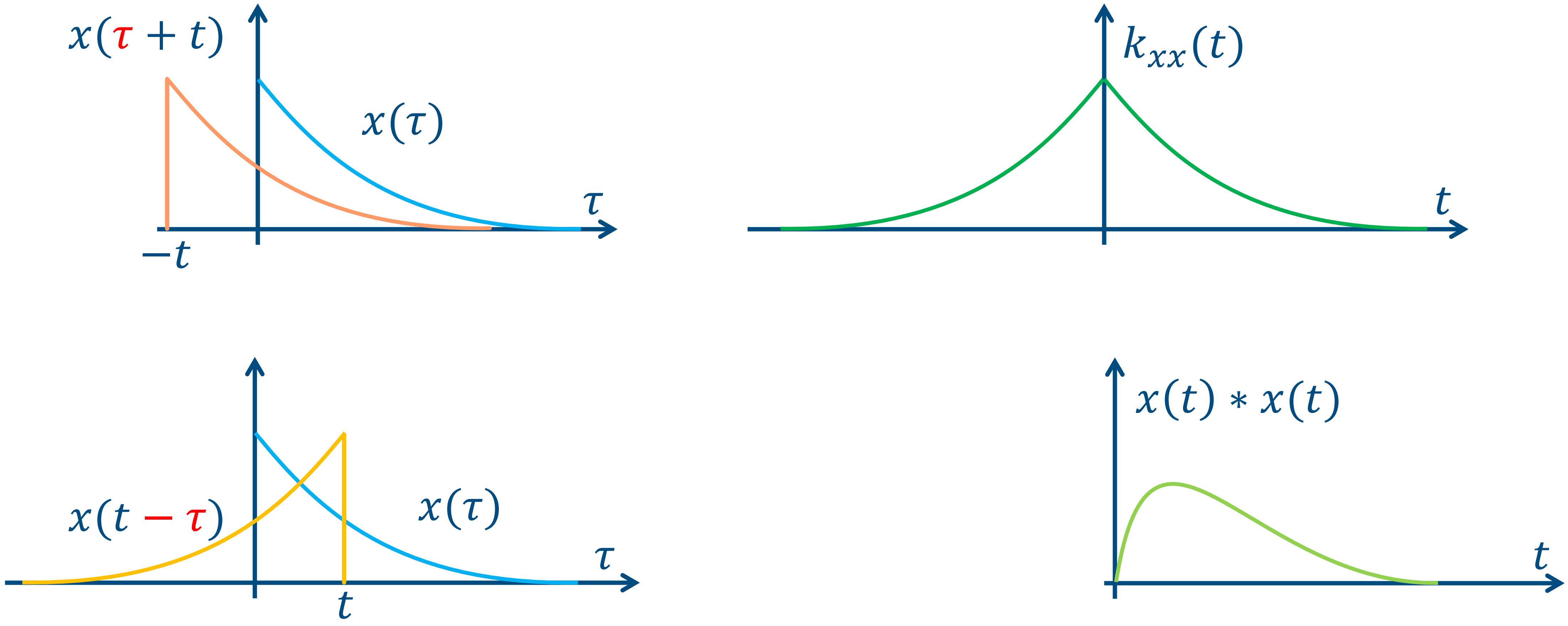
Example: rectangular pulse





$$k_{xx}(\tau) = \int x(t)x(t+\tau)dt = A^2(T - |\tau|)$$

Autocorrelation and convolution



Frequency domain

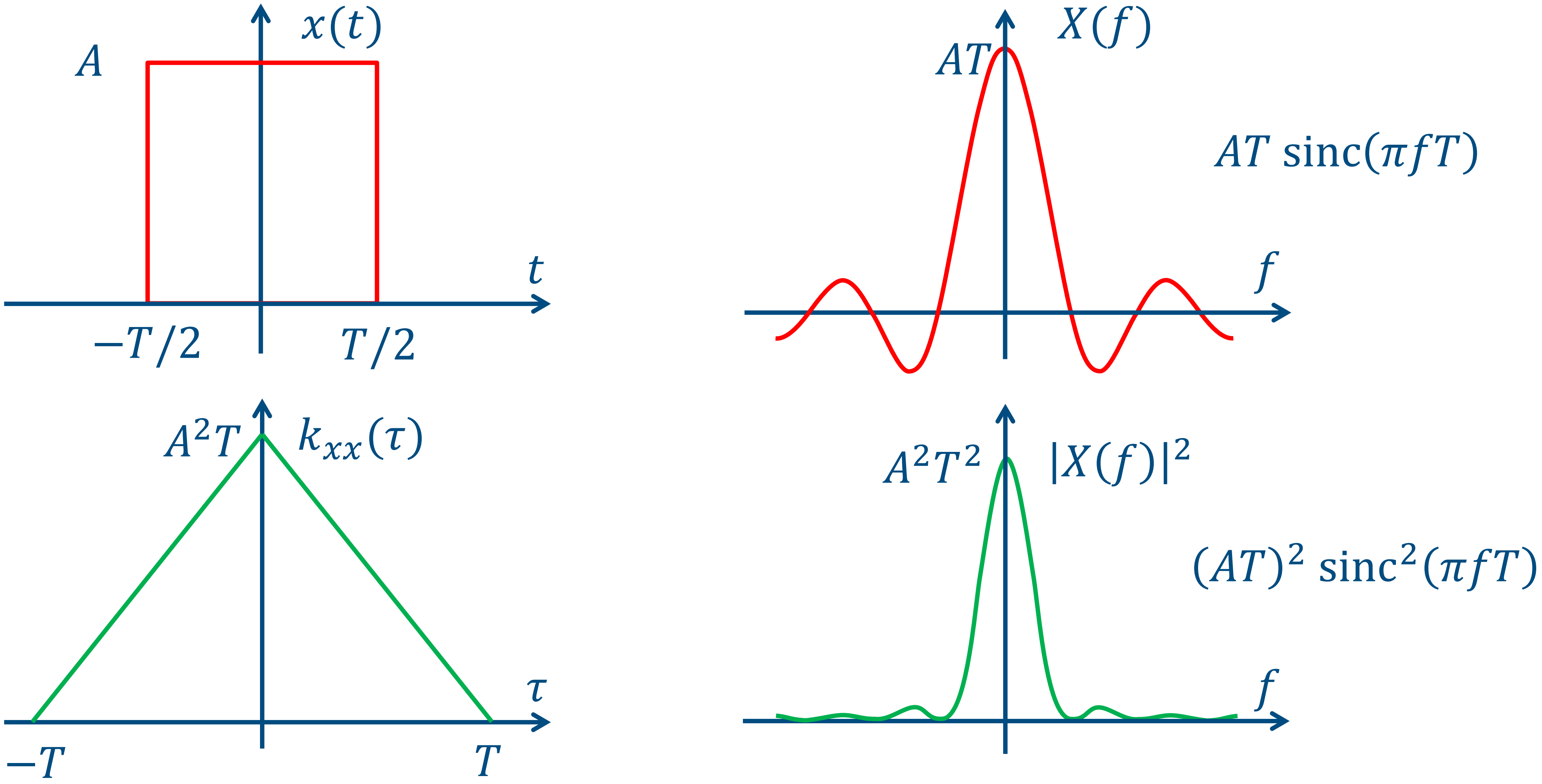
$$k_{xx}(t) = k_{xx}(-t) = \int x(\tau)x(\tau - t)d\tau = x(t) * x(-t)$$

$$\mathcal{F}[k_{xx}(t)] = X(f)X^*(f) = |X(f)|^2$$ (Real and even)

$$E = \int x^2(t)dt = k_{xx}(0) = \int |X(f)|^2 df$$ Also from Parseval theorem

$|X(f)|^2$ is called the energy spectral density

Example: rectangular pulse



Power signals

- These are signals not belonging to L$^2(\mathcal{R})$, whose energy diverges (e.g., periodic signals)
- We then consider the truncated (energy) signal

$$x_T(t) = \begin{cases} x(t) & \forall |t| \leq T \\ 0 & \forall |t| > T \end{cases}$$

for which we can define the Fourier transform $X_T(f)$ and the autocorrelation

$$k_{xx}^T(\tau) = \int x_T(t) x_T(t+\tau) dt\,,$$

where $\mathcal{F}\left[k_{xx}^T(\tau)\right] = |X_T(f)|^2$

Autocorrelation and PSD

$$K_{xx}(\tau) = \lim_{T\to\infty} \frac{1}{2T} k_{xx}^{T}(\tau) = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} x(t)x(t+\tau)dt$$

$$\mathcal{F}[K_{xx}(\tau)] = \lim_{T\to\infty} \frac{1}{2T} |X_T(f)|^2 = S(f)$$

$$P = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} x^2(t)dt = K_{xx}(0) = \int S(f)df$$

signal power

power spectral density

Example: sinusoidal signal

$$x(t) = B\cos\omega_r t$$

$$K_{xx}(\tau) = \lim_{T\to\infty}\frac{B^2}{2T}\int_{-T}^{T}\cos\omega_r t\cos\omega_r(t+\tau)\,dt$$

$$= \lim_{T\to\infty}\frac{B^2}{2T}\int_{-T}^{T}\cos\omega_r t\,(\cos\omega_r t\cos\omega_r\tau - \sin\omega_r t\sin\omega_r\tau)dt =$$

$$B^2\cos\omega_r\tau\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\cos^2\omega_r t\,dt = \frac{B^2}{2}\cos\omega_r\tau$$

Frequency domain

$$x_T(t) = B\cos\omega_r t \;\; \mathrm{rect}(-T,T)$$

$$X_T(f) = \frac{B}{2}\big(\delta(f-f_r)+\delta(f+f_r)\big) * 2T\mathrm{sinc}(2\pi f T)$$

$$= \frac{B}{2}(2T)\big(\mathrm{sinc}(2\pi(f-f_r)T)+\mathrm{sinc}(2\pi(f+f_r)T)\big)$$

$$|X_T(f)|^2 = \frac{B^2}{4}(2T)^2\big(\mathrm{sinc}^2(2\pi(f-f_r)T)+\mathrm{sinc}^2(2\pi(f+f_r)T)\big)$$

$$S_x(f) = \lim_{T\to\infty}\frac{1}{2T}|X_T(f)|^2 = \frac{B^2}{4}\big(\delta(f-f_r)+\delta(f+f_r)\big)$$

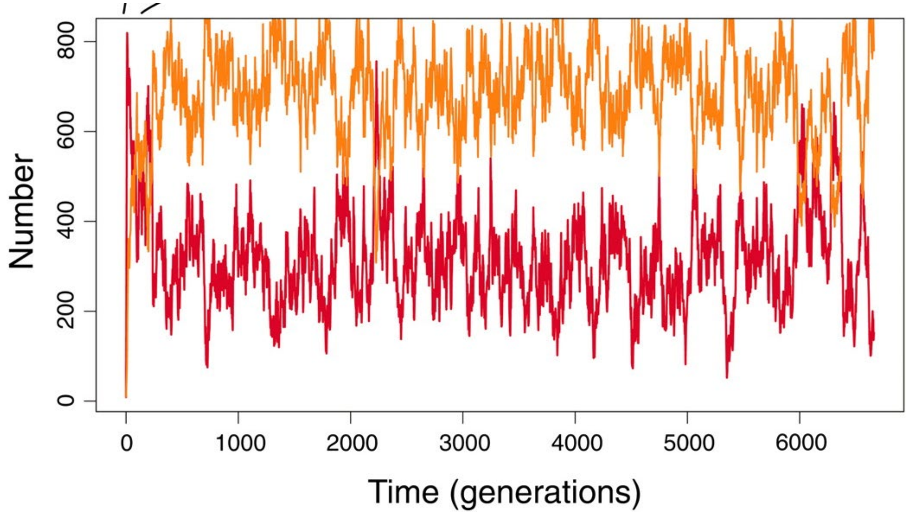
Outline

- Signals in time and frequency domains
- Random processes
- White noise and approximations

Stochastic (or random) processes

- They represent signals that are described in probabilistic terms
- A few examples
  - Current flowing through devices
  - Wireless signal received by mobile phones
  - Waiting time at bus stops
  - Stock market
  - …

Examples



Demographics

From [1]

Music

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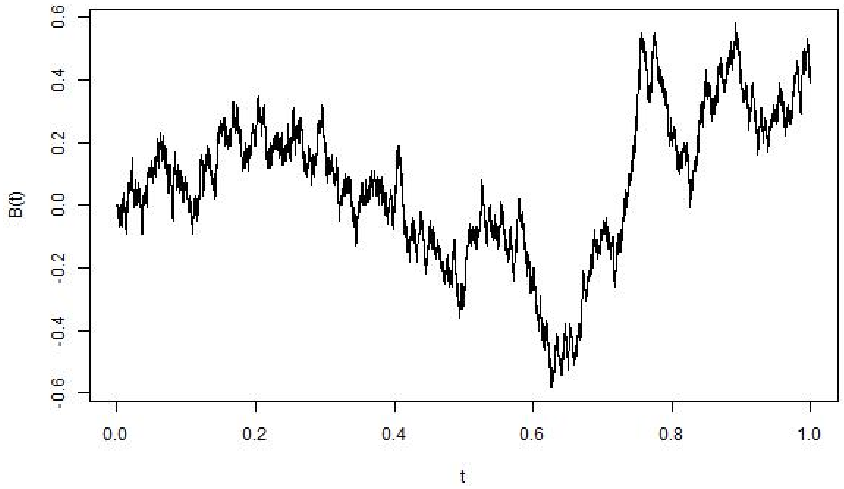
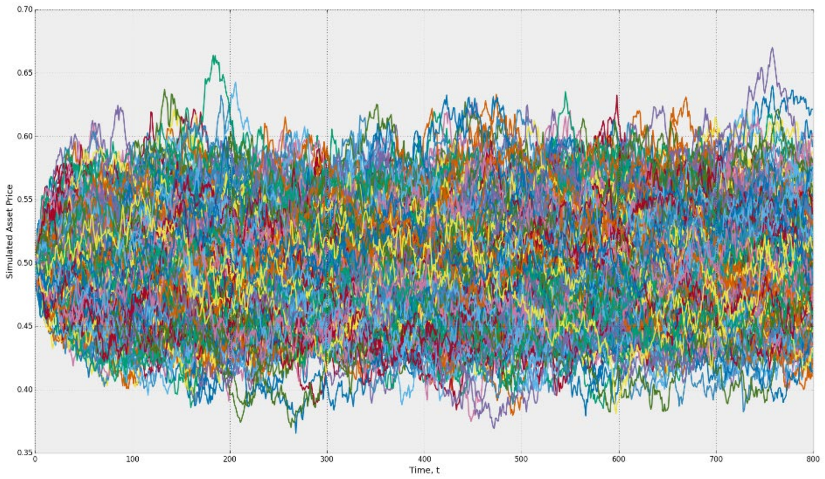




Finance

From [3]

Process realizations

A random process is defined by the probability density function $p(x, t)$ or by the joint probabilities $p(x_1, \dots, x_n; t_1, \dots, t_n)$

- At any given time, the process becomes a random variable
- For any random variable (i.e., experiment), the process becomes a deterministic function of time, called realization or sample

Mean and variance

- Mean value

$$\overline{x(t)} = \int xp(x;t)dx$$

- Mean square value

$$\overline{x^2(t)} = \int x^2 p(x;t)dx$$

- Variance

$$\sigma_x^2(t) = \int (x - \bar{x})^2 p(x;t)dx = \overline{x^2} - \bar{x}^2$$

Moment functions

- Autocorrelation

$$R_{xx}(t_1,t_2) = \overline{x(t_1)x(t_2)} = \iint x(t_1)x(t_2)\,p(x_1,x_2;t_1,t_2)dx_1dx_2$$

- Autocovariance

$$C_{xx}(t_1,t_2) = \overline{\left(x(t_1) - \overline{x(t_1)}\right)\left(x(t_2) - \overline{x(t_2)}\right)}$$

$$= \iint (x_1 - \overline{x_1})(x_2 - \overline{x_2})p(x_1,x_2;t_1,t_2)dx_1dx_2$$

Dependent on t_1 and t_2

Example: random lines

$x(t) = A + Bt,\ t \geq 0$, where A and B are independent random variables

$$R_{xx}(t_1, t_2) = \overline{x(t_1)x(t_2)} = \overline{(A + Bt_1)(A + Bt_2)} = \overline{A^2} + \overline{AB}(t_1 + t_2) + \overline{B^2}t_1 t_2$$

$$C_{xx}(t_1, t_2) = \sigma_A^2 + \sigma_B^2 t_1 t_2$$

Stationary processes

- The joint probability distributions are independent of the time shift

$$p(x_1, \dots, x_n; t_1, \dots, t_n) = p(x_1, \dots, x_n; t_1 + T, \dots, t_n + T) \quad \forall\, n, t, T$$

- The joint pdfs become
  - $p(x, t) = p(x, t + T)\ \forall t, T \Rightarrow p(x, t) = p(x) \Rightarrow \bar{x}$ and σ_x^2 are independent of time
  - $p(x_1, x_2; t_1, t_2) = p(x_1, x_2; t_1, t_1 + \tau)\ \forall t_1 \Rightarrow p(x_1, x_2; t_1, t_2) = p(x_1, x_2; \tau)$

Moment functions (stationary case)

- Autocorrelation

$$R_{xx}(\tau) = \overline{x(t_1)x(t_1+\tau)} = \iint x_1 x_2 p(x_1, x_2; \tau) dx_1 dx_2$$

- Autocovariance

$$C_{xx}(\tau) = \overline{\left(x(t_1) - \overline{x(t_1)}\right)\left(x(t_2) - \overline{x(t_2)}\right)}$$

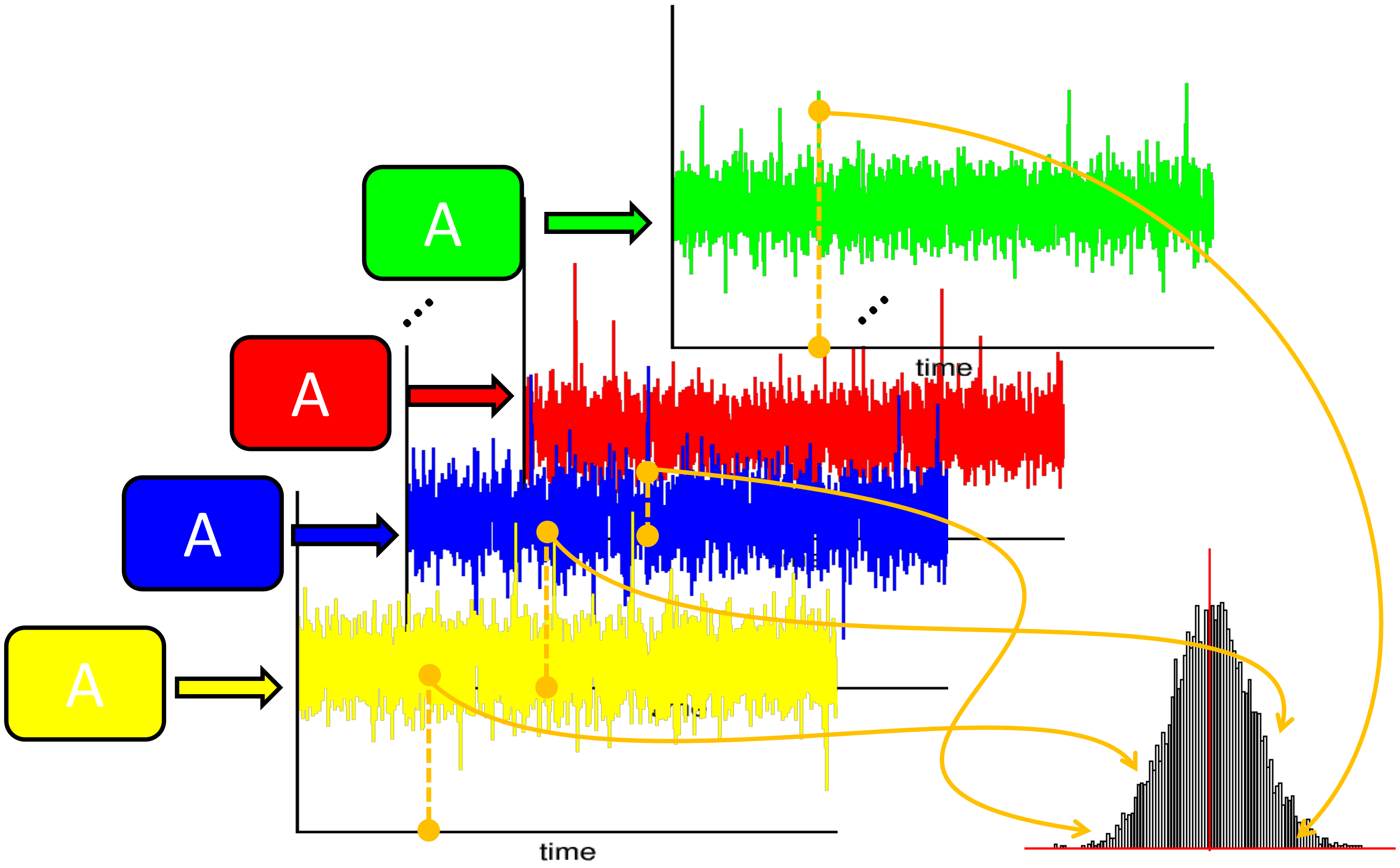
$$= \iint (x_1 - \bar{x})(x_2 - \bar{x}) p(x_1, x_2; \tau) dx_1 dx_2$$

Only dependent on $\tau = t_2 - t_1$

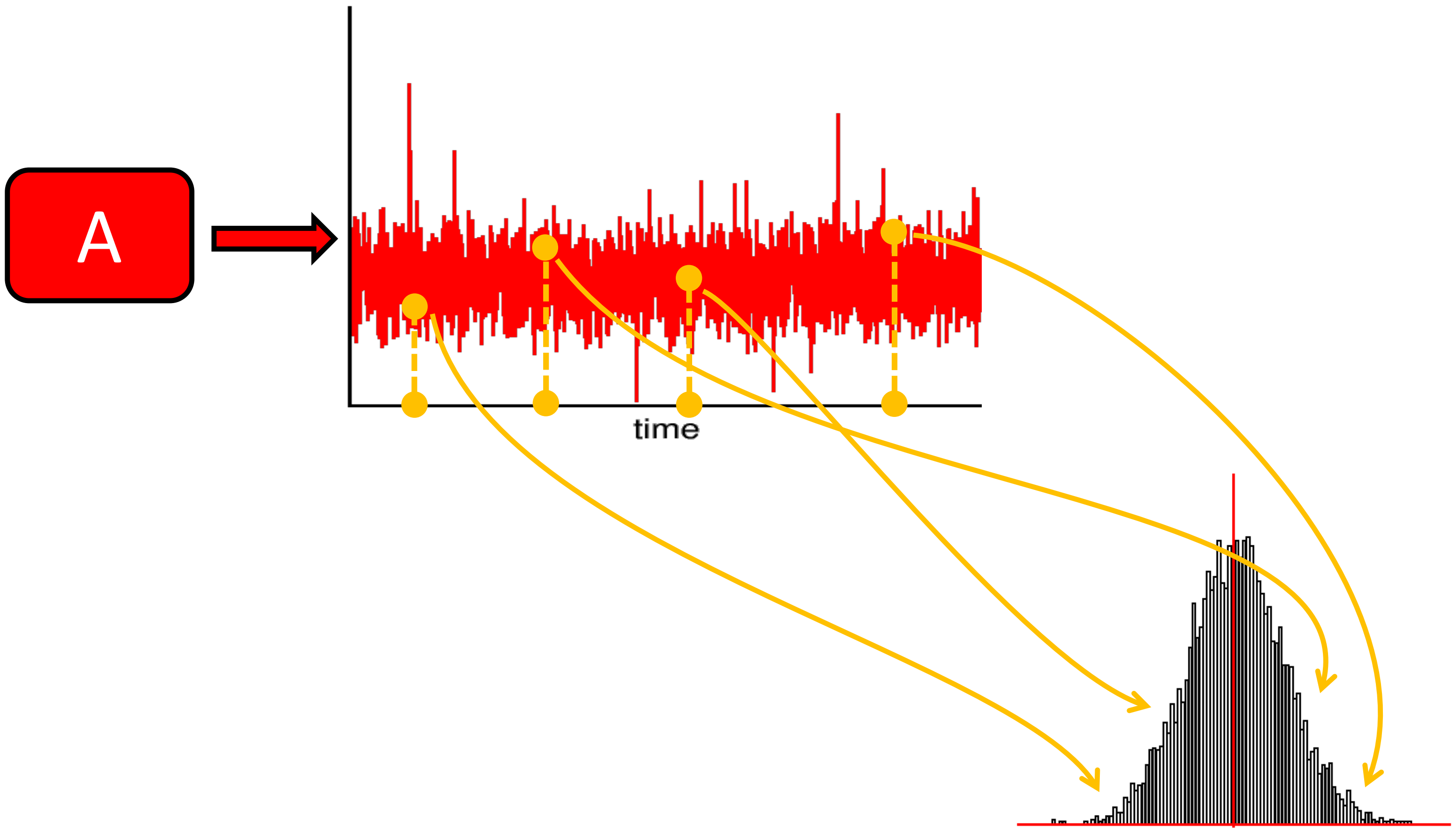
A few notes

- In the general case, $C_{xx} = R_{xx} - \overline{x_1}\,\overline{x_2}$
- Values at $\tau = 0$
  - $R_{xx}(0) = \overline{x^2}$
  - $C_{xx}(0) = \sigma_x^2$
- We will mainly consider processes with zero mean value $\Rightarrow$ $R_{xx} = C_{xx}$

Ensemble averages



Temporal averages



Ergodic processes

$$\langle x \rangle = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} x(t)dt = \bar{x}$$

$$K_{xx}(\tau) = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} x(t)x(t+\tau)dt = R_{xx}(\tau)$$

- Temporal statistics converge to the ensemble ones, for all moments
- Ergodic processes are stationary (the inverse is not strictly true – see Appendix I – but often verified in practice)

Frequency domain

- Any single realization $x_i(t)$ is a power signal, and we can define

$$S_i(f) = \lim_{T\to\infty} \frac{1}{2T} \left|X_{T,i}(f)\right|^2$$

- $S_i(f)$ is a random variable. We now take the ensemble average

$$S(f) = \overline{S_i(f)} = \lim_{T\to\infty} \frac{1}{2T} \overline{\left|X_{T,i}(f)\right|^2}$$

- $S(f)$ is called the power spectral density of the random process
- $\mathcal{F}^{-1}[S(f)] = R_{xx}(\tau)$ (Wiener-Khinchin theorem, see Appendix II)

Example: random phase sinusoid

- Random process $x(t) = B\cos(\omega_r t + \phi)$ where ϕ is a random variable uniformly distributed in $[-\pi, \pi] \Rightarrow \bar{x} = \langle x \rangle = 0$
- Ensemble autocorrelation of the process is

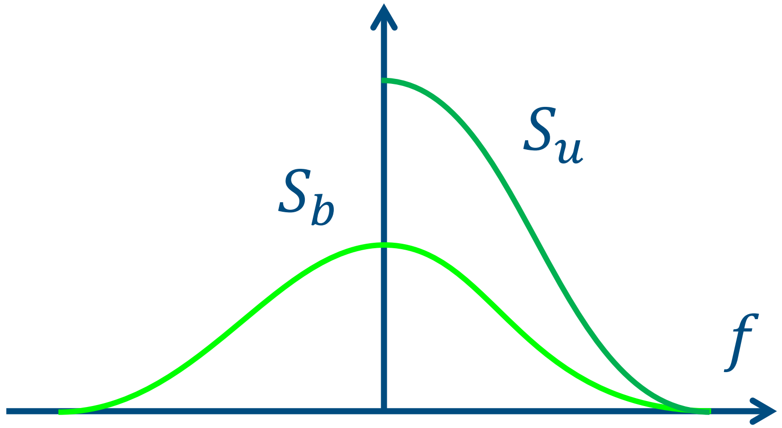
$$R_{xx}(\tau) = B^2 \int_{-\pi}^{\pi} \cos(\omega_r t + \phi)\cos(\omega_r (t + \tau) + \phi) p(\phi) d\phi$$

$$= \frac{B^2}{2} \cos \omega_r \tau = K_{xx}(\tau)$$

- PSD becomes then $S(f) = \frac{B^2}{4}\left(\delta(f - f_r) + \delta(f + f_r)\right)$

Uni- and bilateral spectra

- $S(f)$ is real and even, extending from $-\infty$ to $+\infty \Rightarrow$ bilateral power spectrum
- In circuit calculations, positive frequencies only are of interest $\Rightarrow$ a unilateral PSD is defined, extending from $f = 0$ to $+\infty$
- $S_u(f) = 2S_b(f)\ \forall f \geq 0$



Outline

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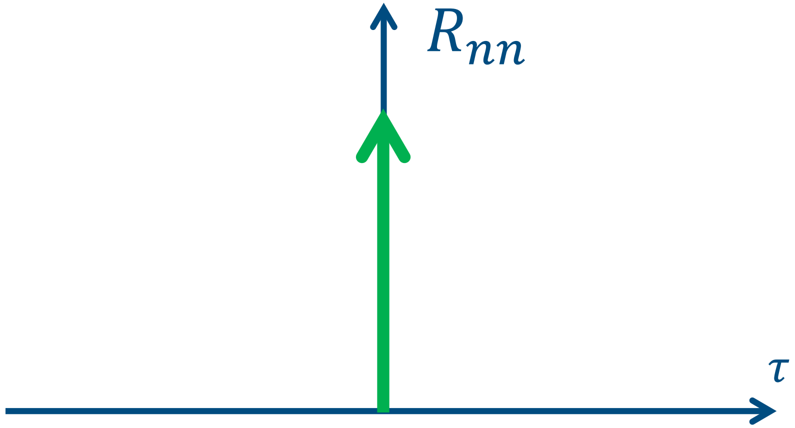
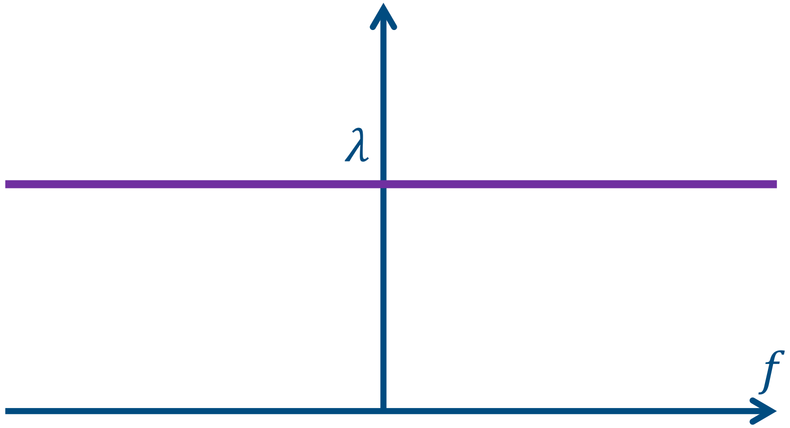
- Random processes
- White noise and approximations

White noise

$$R_{nn}(\tau) = \lambda\delta(\tau)$$

$$S_n(f) = \lambda$$





The process is totally uncorrelated with itself

$$\overline{n^2} = \infty$$

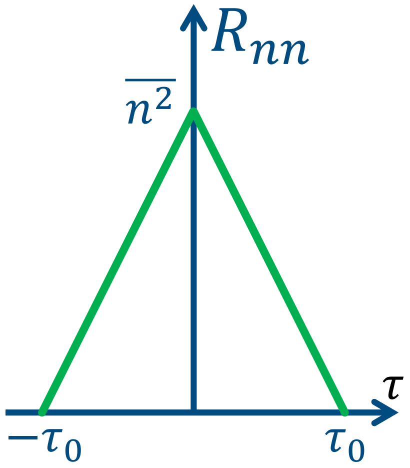
White noise approximations

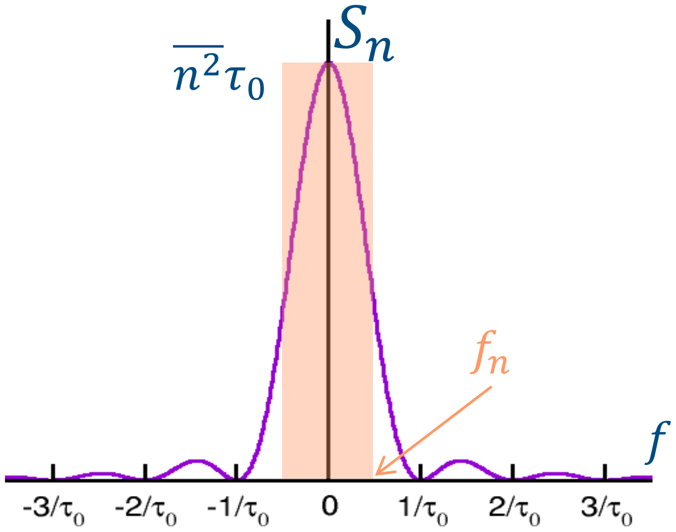
- Real noises are only approximately white, i.e., they behave as such in a limited spectral (or time) range
- Correlation time and bandwidth are related by the time-bandwidth relation:

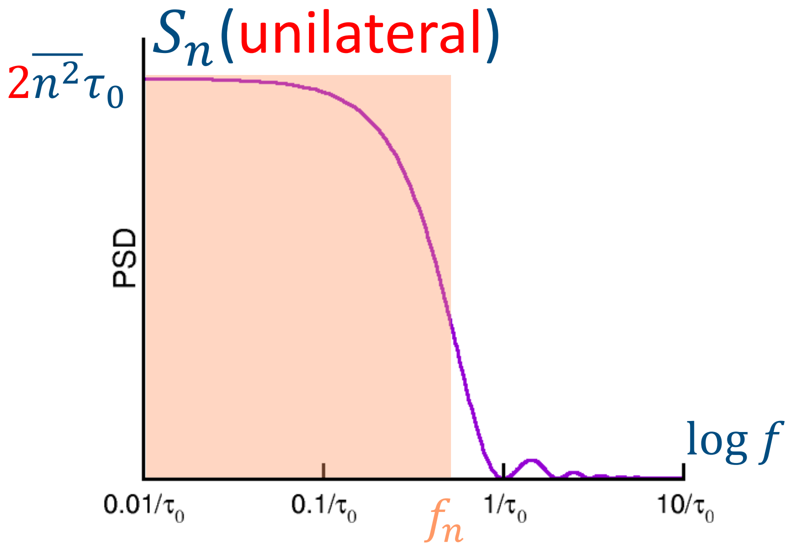
$$R_{nn}(\tau) = \lambda g(\tau) \qquad g(\tau) \approx 0 \;\; \forall \, |\tau| > \tau_0$$

$$S_n(f) \approx \text{constant} \qquad \forall \, |f| < 1/\tau_0$$

Triangular approximation



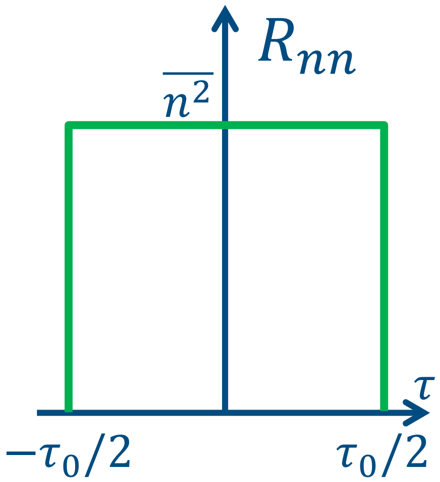


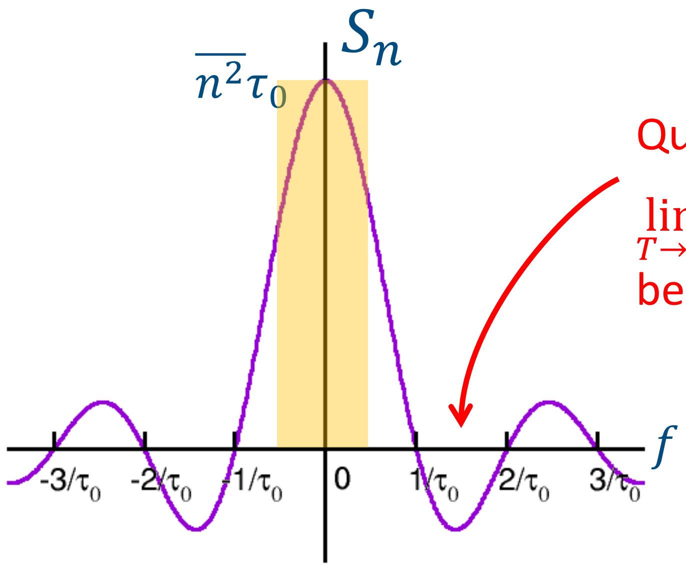


$$S_n(f) = \overline{n^2}\tau_0 \left(\frac{\sin(\pi f \tau_0)}{\pi f \tau_0}\right)^2$$

$$\overline{n^2} = \int S_n(f)df = S_n(0)2f_n \Rightarrow f_n = \frac{1}{2\tau_0}$$

Rectangular approximation





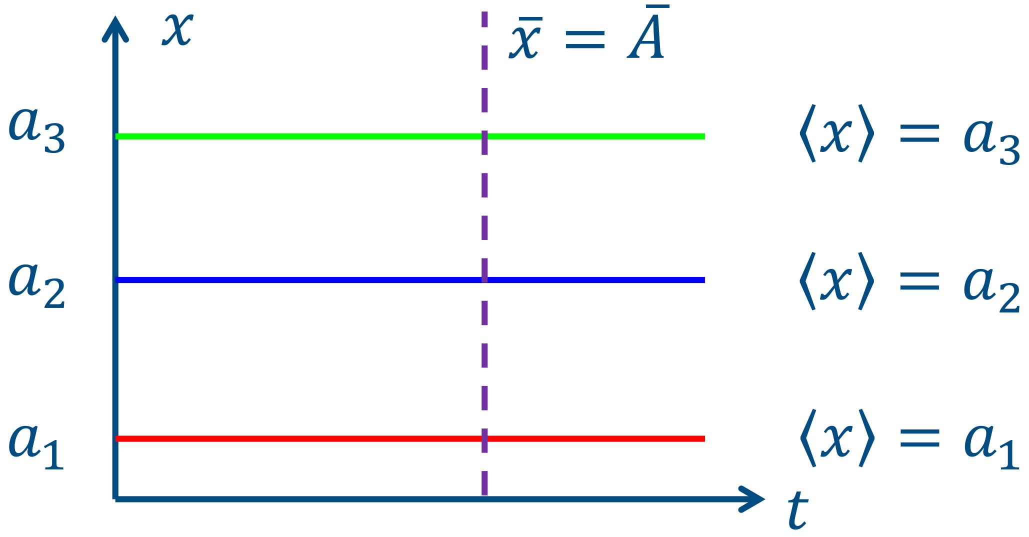
Question: given that $S(f) = \lim_{T\to\infty} \frac{1}{2T} \overline{|X_{T,i}(f)|^2}$, how can it be negative?

$$S_n(f) = \overline{n^2}\tau_0 \frac{\sin(\pi f \tau_0)}{\pi f \tau_0}$$

$$\overline{n^2} = \int S_n(f)df = S_n(0)2f_n \Rightarrow f_n = \frac{1}{2\tau_0}$$

Appendix I: a simple counter-example

- We consider the process $x(t) = A$, random variable in [0,1]



- $x(t)$ is stationary but not ergodic

Appendix II: Wiener-Khinchin theorem

$$\mathcal{F}^{-1}[S(f)] = \int \lim_{T\to\infty} \frac{1}{2T} \overline{|X_T(f)|^2}\, e^{j2\pi f\tau} df = \lim_{T\to\infty} \frac{1}{2T} \int \overline{X_T(f) X_T^*(f)}\, e^{j2\pi f\tau} df =$$

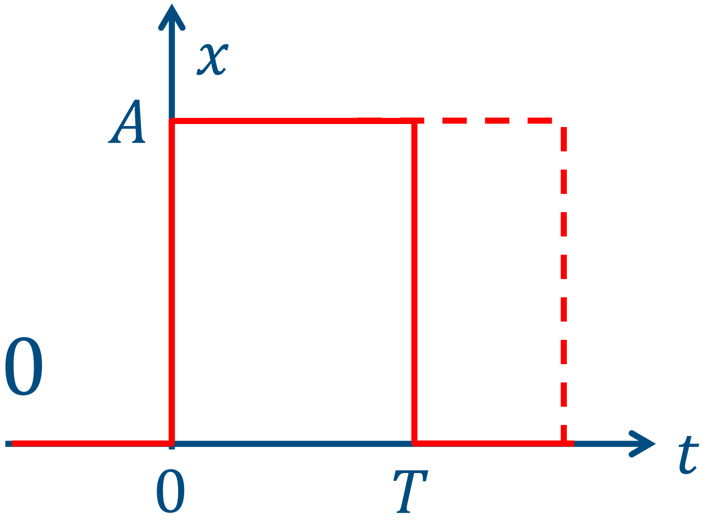
$$= \lim_{T\to\infty} \frac{1}{2T} \int \overline{\int_{-T}^{T} x(t_1) e^{-j2\pi f t_1} dt_1 \int_{-T}^{T} x(t_2) e^{j2\pi f t_2}\, dt_2}\, e^{j2\pi f\tau} df =$$

$$= \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \int_{-T}^{T} \overline{x(t_1) x(t_2)} \int e^{j2\pi f(t_2 - t_1 + \tau)} df\, dt_1 dt_2 =$$

$$= \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \int_{-T}^{T} R_{xx}(t_1, t_2) \delta(\tau + t_2 - t_1) dt_1 dt_2 = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} R_{xx}(\tau) dt_1 = R_{xx}(\tau)$$

Appendix III: more examples

- $x(t)$ = rectangular pulse starting at $t = 0$ and random width T, with $p(T) = \lambda e^{-\lambda T}$ (Poisson process)
- Definitely non-stationary!



$$\langle x \rangle = 0; \quad \overline{x(t)} = A \int_t^{\infty} p(T)dT = Ae^{-\lambda t}, t > 0$$

$$R_{xx}(t_1, t_2) = 0 \; \forall \; t_1 < 0 \cup t_2 < 0$$

$$R_{xx}(t_1, t_2) = A^2 \int_t^{\infty} p(T)dT = A^2 e^{-\lambda t}, t = \max(t_1, t_2) \; \forall \; t_1, t_2 \geq 0$$

References

1. http://www.genetics.org/content/185/4/1345
2. https://www.ams.jhu.edu/dan-mathofmusic/stochastic-processes/
3. http://www.turingfinance.com/random-walks-down-wall-street-stochastic-processes-in-python/