Electronics - Tutorage 4

David G. Refaldi, davidgianluigi.refaldi@polimi.it

9^{th} May, 2025

Noise in Linear Time-Variant (LTV) systems

Dealing with noise in OA circuits we exploited the theory on Linear Time-Invariant (LTI) systems to understand how noise is transferred from the input to the output of an OA stage. Nevertheless, such theory is not able to properly handle the variety of filters which may come in handy during the stage of signal conditioning. For such a reason, a more general theory can be derived for Linear Time-Variant (LTV) systems, *i.e.* systems which have a non-stationary response in time.

In most of the situations it is better to work in the time-domain to solve exercises with LTV systems. Similarly to LTI, LTV systems can be described through a function, called *wighting function* $w(t,\tau)$. The weighting function corresponds to the system response to a pulse centered at time τ at the instant t (note that, from this definition, $w(t,\tau) = 0$ if $t < \tau$). Exploiting this function, the output of a LTV system subject to the input signal $x(t)$ can be written as:

$$y(t) = \int x(\tau) \cdot w(t,\tau) \cdot d\tau \tag{1}$$

Let's now understand how noise is transferred through these kind of systems. For a general non-stationary random process at the input of the system, it is possible to define it's autocorrelation, which will depend on both instants at which it is evaluated, as $R_{xx}(t_1,t_2)$. The output noise will be characterized by a non-stationary autocorrelation given by:

$$R_{yy}(t_1,t_2) = \iint R_{xx}(\alpha,\beta) \cdot w(t_1,\alpha) \cdot w(t_2,\beta) \cdot d\alpha d\beta \tag{2}$$

While the rms value of the noise can be evaluated simply by taking $t_1 = t_2 = t$ in equation (2), obtaining:

$$\overline{n_y(t)^2} = R_{yy}(t,t) = \iint R_{xx}(\alpha,\beta) \cdot w(t,\alpha) \cdot w(t,\beta) \cdot d\alpha d\beta \tag{3}$$

Which clearly shows that the output rms noise will be non-stationary. This last property is caused by the time-variant nature of the filter. Indeed, if we consider a stationary random process, for which the autocorrelation depends only on the reciprocal time difference between the instants at which we compute it, we obtain:

$$R_{yy}(t_1,t_2) = \iint R_{xx}(\beta-\alpha) \cdot w(t_1,\alpha) \cdot w(t_2,\beta) \cdot d\alpha d\beta = \int R_{xx}(\gamma) \int w(t_1,\alpha) \cdot w(t_2,\alpha+\gamma) \cdot d\alpha d\gamma \tag{4}$$

Assuming then to evaluate the output rms noise, equation (4) becomes:

$$\overline{n_y(t)^2} = R_{yy}(t,t) = \int R_{xx}(\gamma) \int w(t,\alpha) \cdot w(t,\alpha+\gamma) \cdot d\alpha d\gamma = \int R_{xx}(\gamma) \cdot k_{w_{tt}}(\gamma) \cdot d\gamma \tag{5}$$

Where we defined $k_{w_{tt}}(\tau)$ the weighting function time-correlation.

Exploiting the properties of Fourier transforms (in particular, Parseval's theorem), it is possible to work out equivalent expressions in the frequency domain (transforming with respect to the time variable τ).

Low Pass Filter (LPF)

Low pass filters are typically LTI systems which can help in conditioning low frequency signals subjected to high-frequency noise. A classic application is the one of processing constant signals with a superimposed white noise. In general, the transfer function of a low-pass filter can be written as:

$$H(f) = \frac{1}{1 + sT_F} \tag{6}$$

With an associated impulse-response given by:

$$h(t) = \frac{1}{T_F} e^{-t/T_F} u(t) \tag{7}$$

Note that, as LPFs are LTI systems, the weighting function $w(t, \tau)$ is nothing more than the pulse-response shifted and reversed $h(\tau - t)$. If we assume that the input noise is stationary, we can proceed to calculate the output noise autocorrelation as:

$$R_{yy}(\tau) = R_{xx}(\tau) * k_{hh}(\tau) = R_{xx}(\tau) * \int h(t) \cdot h(t + \tau) \cdot dt \tag{8}$$

Where the time-correlation of the weighting function $k_{hh}(\tau)$ assumes the expression of:

$$k_{hh}(\tau) = \frac{1}{2T_F} e^{-|\tau|/T_F} \tag{9}$$

Let's now consider the specific case of white noise, with an input autocorrelation given by $R_{xx}(\tau) = \lambda \cdot \delta(\tau)$. Then, the output rms noise is easily evaluated as:

$$\overline{n_y^2} = R_{yy}(0) = \left[\lambda \cdot \delta(\tau) * \frac{1}{2T_F} e^{-|\tau|/T_F}\right]_{\tau=0} = \frac{\lambda}{2T_F} \tag{10}$$

Note that the longer is T_F the smaller are the output fluctuations. This is easy to understand, as a larger T_F results is a longer averaging operation. The longer the averaging is conducted, the more the output of the random process will converge to its mean value, which is zero. This could have been understood also from the frequency-domain perspective, given the bilater PSD of white noise $S_x(f) = \lambda$. Indeed, the output rms noise can be written as:

$$\overline{n_y^2} = \int \lambda \cdot |H(f)|^2 = \int \lambda \cdot \frac{1}{1 + (2\pi f T_F)^2} \cdot df = \frac{\lambda}{2\pi T_F} \int \frac{1}{1 + x^2} \cdot dx = \frac{\lambda}{2T_F} \tag{11}$$

From the equivalent rectangle approximation, we observe that the white noise equivalent bandwidth of LPF is given by $1/4T_F$, to be compared with the signal bandwidth, given by $1/2\pi T_F$. Note that the bandwidth of the noise is larger than the one of the signal by a term $\pi/2$ ($BW_n = \pi/2BW_s$).

To check the noise conditioning capability of the LPF we can test how it works when the input signal is a step at $t = 0$, *i.e.* $x(t) = Au(t)$. The output signal will be:

$$y(t) = \int Au(\tau) \cdot \frac{1}{T_F} e^{-(t-\tau)/T_F} \cdot d\tau = A(1 - e^{-t/T_F}) \tag{12}$$

Assuming that we evaluate the output when the exponential transient has vanished, we obtain an output signal-to-noise ratio:

$$(S/N)_y = \frac{A}{\sqrt{\lambda}} \sqrt{2T_F} \tag{13}$$

To compare it with the input (S/N) we need to consider a quasi-white noise with equivalent bandwidth $f_n = 1/2T_n$ ($T_F \gg T_n$), that leads to:

$$(S/N)_x = \frac{A}{\sqrt{\lambda}} \sqrt{T_n} \tag{14}$$

Eventually, we observe that after the LPF the signal-to-noise ratio is increased by a factor $\sqrt{\frac{2T_F}{T_n}}$.

Gated Integrator (GI)

The gated integrator (GI) is a simple example of LTV system which can be used to condition fast signals (pulses) with superimposed high-frequency noise (white). Figure 1 reports a simple implementation.

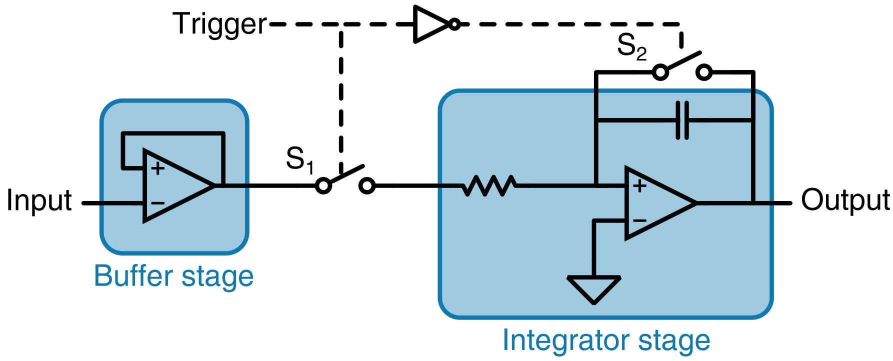

Figure 1: An example of gated integrator implementation.

The working principle is that the signal (eventually buffered) is integrated on a finite time window, which is defined by the control trigger signal. Indeed, the trigger switches S_1 and S_2 in a way that only one of them is closed at the same time. When S_1 is closed (and S_2 is open), the buffered signal is integrated at the output, while when S_1 is open (and S_2 is closed), the feedback capacitor is discharged and the output is forced to GND.
Let's assume that the GI is programmed to open a single integration window in the time interval $[0, t]$. Then the weighting function can be easily obtained from the following reasoning: if a pulse arrives at $\tau < 0$ or at $\tau > t$ the integration window will be closed, thus the output at any time will be zero. Instead, if a pulse arrives during the integration window, *i.e.* $0 < \tau < t$, the signal will be integrated to the output. As we know that the integral of the pulse is the step function, we immediately get that at any reading time such pulse will result in a constant signal equal to the gain K of the GI. Eventually we obtain:

$$w(t, \tau) = K(u(\tau) - u(\tau - t)) \tag{15}$$

For any generic input signal $x(t)$ we get at the output a signal given by:

$$y(t) = \int x(\tau) \cdot w(t, \tau) \cdot d\tau = K \int_0^t \frac{t}{t} x(\tau) \cdot d\tau = Kt\langle x(t)\rangle \tag{16}$$

Indeed, the output signal is proportional to the time average of the input signal during the integration window.
Assuming now to have an input white noise of autocorrelation $R_{xx}(\tau) = \lambda \cdot \delta(\tau)$ we can calculate how it is transferred to the output by first computing the weighting function time-correlation $k_{w_{tt}}(\tau)$:

$$k_{w_{tt}}(\tau) = \begin{cases} 0 & \text{if } |\tau| > t \\ K^2(t + \tau) & \text{if } -t < \tau < 0 \\ K^2(t - \tau) & \text{if } 0 < \tau < t \end{cases} \tag{17}$$

Then, the output rms noise is obtained as:

$$\overline{n_y^2} = \int \lambda \cdot \delta(\gamma) \cdot k_{w_{tt}}(\gamma) \cdot d\gamma = \lambda K^2 t \tag{18}$$

The GI equivalent noise bandwidth is thus given by $1/2t$. To compute the (S/N) let's assume to have a constant input signal of amplitude A during the integration window (so that the output is KAt), obtaining:

$$(S/N)_y = \frac{A}{\sqrt{\lambda}} \sqrt{t} \tag{19}$$

Considering the quasi-white noise at the input (with bandwidth given by $f_n = 1/2T_n$), we get:

$$(S/N)_y = (S/N)_x \cdot \sqrt{\frac{t}{T_n}} \tag{20}$$

Boxcar Averager (BA)

The boxcar averager is an LTV system which find applications in the conditioning of repetitive fast signals (pulses) with superimposed high-frequency noise (white) or in cases where noise is slightly correlated. Figure 2 reports a simple implementation.

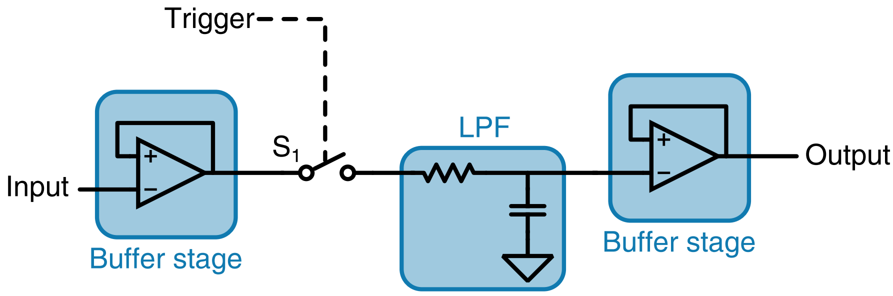

Figure 2: An example of boxcar averager implementation.

The filter works as a LPF when the trigger closes the switch S_1. Instead, when S_1 is open the capacitor has no way of discharging and the output signal is kept constant at the capacitor voltage. The trigger signal is usually a square wave that keeps S_1 open for a period T_O and closed for a period T_C. The weighting function can be shown to be the one reported in Figure 3

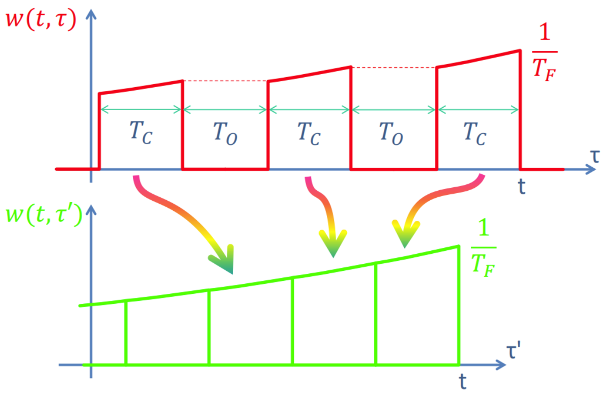

Figure 3: (Top) weighting function of the BA. (Bottom) Equivalent-time weighting function of the BA.

From the standpoint of the equivalent time τ' the weighting function is the same as the one of a LPF, so that we can exploit some known results. In particular, the output rms noise, assuming an input white noise, is given by $\overline{n_y^2} = \lambda/(2T_F)$, as for a LPF. Assuming an input signal of amplitude A, which is constant during the filtering phases we obtain that the output is exactly equal to A and the $(S/N)_y$ is, once again, the one of a LPF. If the signal (of amplitude A) is sufficiently stable over a single filtering window, we can define the single-pulse quantities:

$$y_{sp}(t) = A\int_{t-T_C}^{t} \frac{1}{T_F} e^{-\frac{t-\tau}{T_F}} \cdot d\tau = A\int_{-T_C}^{0} \frac{1}{T_F} e^{-\frac{\gamma}{T_F}} \cdot d\gamma = A\left(1-e^{-\frac{T_C}{T_F}}\right) \tag{21}$$

$$\overline{n_{y,sp}^2} = \lambda k_{wtt}(0) = \lambda \int_{t-T_C}^{t} w(t,\tau)^2 d\tau = \lambda \int_{t-T_C}^{t} \frac{1}{T_F^2} e^{-\frac{2(t-\tau)}{T_F}} d\tau = \frac{\lambda}{2T_F}\left(1-e^{-\frac{2T_C}{T_F}}\right) \tag{22}$$

From which it is possible to observe that the signal-to-noise ratio can assume the expression:

$$(S/N)_y = \frac{A}{\sqrt{\lambda}}\sqrt{2T_F} = (S/N)_{y,sp}\frac{\sqrt{1-e^{-\frac{2T_C}{T_F}}}}{1-e^{\frac{T_C}{T_F}}} = (S/N)_{y,sp}\sqrt{N_{eq}} \tag{23}$$

Where we defined the equivalent number of pulses N_{eq} to be:

$$N_{eq} = \frac{1-e^{-\frac{2T_C}{T_F}}}{\left(1-e^{\frac{T_C}{T_F}}\right)^2} = \frac{1+e^{-\frac{T_C}{T_F}}}{1-e^{\frac{T_C}{T_F}}} \approx 2\frac{T_F}{T_C} \tag{24}$$

Where the last approximation holds if $T_F \gg T_C$.

Ratemeter integrator (RI)

The ratemeter integrator (RI) is a filter conceptually similar to the boxcar averager. Figure 4 reports a simple implementation.

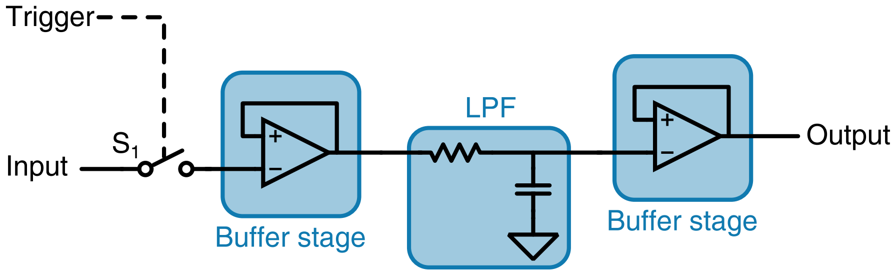

Figure 4: An example of ratemeter integrator implementation.

Similarly to the BA, the filter works with a single switch S_1. When the switch is closed the input signal is trasferred to a buffer, thus being filtered through a LPF. Contrary to the BA, however, when the switch is open the LPF capacitor is free to discharge on it's series resistance directly through the output of the first stage. The weighting function is shown in Figure 5.

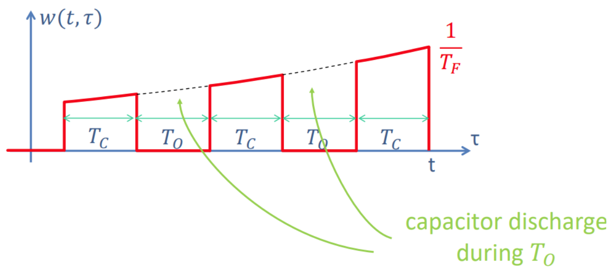

Figure 5: Weighting function of the ratemeter integrator.

To evaluate the performance of the filter we can check how it behaves when fed with a constant signal (or a train of pulses synchronized with the trigger) of amplitude A with some white noise superimposed. The output signal can be computed by summing the contribution of each pulse $y^n(t)$ (assuming $t=0$):

$$y^n(t) = A\int_{-n(T_C+T_O)-T_C}^{-n(T_C+T_O)} \frac{e^{\tau/T_F}}{T_F}d\tau = A\left(1-e^{-\frac{T_C}{T_F}}\right)e^{-n\frac{T_C+T_O}{T_F}} \tag{25}$$

$$y(t) = \sum_0^\infty y^n(t) = A\left(1-e^{-T_C/T_F}\right)\sum_0^\infty e^{-n\frac{T_C+T_O}{T_F}} = A\frac{1-e^{-\frac{T_C}{T_F}}}{1-e^{-\frac{T_C+T_O}{T_F}}} \tag{26}$$

For what concern the noise, we obtain:

$$k^n_{w_{tt}}(0) = \int_{-n(T_C+T_O)-T_C}^{-n(T_C+T_O)} \frac{e^{2\tau/T_F}}{T_F^2}d\tau = \frac{1-e^{-2\frac{T_C}{T_F}}}{2T_F}e^{-2n\frac{T_C+T_O}{T_F}} \tag{27}$$

$$k_{w_{tt}}(0) = \sum_0^\infty k^n_{w_{tt}}(0) = \frac{1}{2T_F}\frac{1-e^{-2\frac{T_C}{T_F}}}{1-e^{-2\frac{T_C+T_O}{T_F}}} \tag{28}$$

The resulting $(S/N)_y$ is then:

$$(S/N)_y = \frac{A}{\sqrt{\lambda}}\sqrt{T_F}\cdot\frac{1-e^{-\frac{T_C}{T_F}}}{\sqrt{1-e^{-2\frac{T_C}{T_F}}}}\cdot\frac{\sqrt{1-e^{-2\frac{T_C+T_O}{T_F}}}}{1-e^{-\frac{T_C+T_O}{T_F}}} = (S/N)^{BA}_{sp}\cdot\sqrt{N_{eq}} \tag{29}$$

Where we recognized the expression of the (S/N) of the single-pulse BA multiplied by a term associated to the equivalent number of pulses N_{eq}, which in this case can be expressed as:

$$N_{eq} = \frac{1 - e^{-2\frac{T_C+T_O}{T_F}}}{\left(1 - e^{-\frac{T_C+T_O}{T_F}}\right)^2} = \frac{1 + e^{-\frac{T_C+T_O}{T_F}}}{1 - e^{-\frac{T_C+T_O}{T_F}}} \approx \frac{2T_F}{T_C + T_O} \tag{30}$$

Where the last approximation holds if $T_F \gg T_C + T_O$

List of relevant exams

The following exams contain questions related to the filters studied in this tutorage:

- Exam of 10$^{\text{th}}$ July 2015, exercise 2
- Exam of 25$^{\text{th}}$ September 2015, exercise 2
- Exam of 4$^{\text{th}}$ July 2016, exercise 2
- Exam of 21$^{\text{th}}$ July 2016, exercise 2
- Exam of 15$^{\text{th}}$ February 2017, exercise 2
- Exam of 6$^{\text{th}}$ July 2017, exercise 2
- Exam of 20$^{\text{th}}$ July 2017, exercise 2
- Exam of 21$^{\text{st}}$ February 2018, exercise 2
- Exam of 21$^{\text{st}}$ June 2018, exercise 2
- Exam of 20$^{\text{th}}$ July 2018, exercise 2
- Exam of 13$^{\text{th}}$ February 2019, exercise 2
- Exam of 13$^{\text{th}}$ February 2020, exercise 2
- Exam of 18$^{\text{st}}$ June 2020, exercise 2
- Exam of 11$^{\text{th}}$ September 2020, exercise 2
- Exam of 18$^{\text{th}}$ February 2021, exercise 2
- Exam of 23$^{\text{rd}}$ June 2021, exercise 2
- Exam of 21$^{\text{st}}$ January 2022, exercise 2
- Exam of 18$^{\text{th}}$ February 2022, exercise 2
- Exam of 21$^{\text{st}}$ July 2022, exercise 2
- Exam of 20$^{\text{th}}$ January 2023, exercise 2
- Exam of 19$^{\text{th}}$ July 2023, exercise 2
- Exam of 6$^{\text{th}}$ September 2023, exercise 2
- Exam of 7$^{\text{th}}$ February 2024, exercise 2
- Exam of 20$^{\text{th}}$ January 2025, exercise 2
- Exam of 14$^{\text{th}}$ February 2025, exercise 2