Electronics - Tutorage 5

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Discrete-time filters (DTs)

In many situations of interest it is difficult nor impossible to find an analog filter which produces the optimal weighting function for the conditioning of a specific signal. In these situations, the employement of a digital filter, *i.e.* a filter which samples the input signal at some specific and discrete time instants and then performs some arithmetic operations on them, may be a good choice. Indeed, digital filters provide many benefit, among which ease of implementation and re-configurability.
To study how these filters behave we can leverage on the theory of LTV filters. We will assume that the sampling and conversion operations are ideal, *i.e.* sampling captures the instantaneous value of the input signal and the amplitude of the sampled signal is exactly equal to the one of the input. In reality the sampling procedure takes some time, thus the sampled signal is *averaged* on the sampling interval. Moreover, in the digital domain the precision used to represed the sampled signal will be linked to the number of bins of the ADC (Analog-to-Digital Converter). For the purposes of our discussion, however, such problems can be neglected.
Under our assumptions, the weighting function of the discrete-time filter can be seen as an ensemble of Dirac's delta, centered at the sampling instant t_k, *i.e.* $w(t,\tau) = \sum_{k=0}^{N} w_k \delta(\tau - t_k)$, where w_k indicates the weights given to each sample.

Let's go now through some examples of practical importance, to understand how we can proceed in the general case. Let's assume to employ a DT filter to perform a uniform sampling, *i.e.* we take N samples at times t_k, separated by a delay t_s and we weight each one of them by a term $1/N$. Let's assume that out input signal is constant during the sampling, with an amplitude A, then:

$$y(t) = \int x(\tau) \cdot w(t,\tau) d\tau = \sum_{k=0}^{N-1} \frac{1}{N} x(\tau - t_k) = \frac{1}{N} \cdot \sum_{k=0}^{N-1} A = A \tag{1}$$

Clearly, if the signal is constant, we end up with a result which is exactly the input signal, due to the normalization condition set on the samples weight. To understand how noise is transferred we need to calculate the filter time-correlation. It is easy to demonstrate that this is a set of delta functions of decreasing amplitude. For our particular choice of weights, the time-correlation can be written as:

$$k_{w_{tt}}(\gamma) = \begin{cases} 0 & \text{if } \gamma \neq nt_s, n = 0, \pm 1, \pm 2, ... \\ \frac{1}{N^2}(N-n)\delta(\gamma - nt_s) & \text{if } \gamma = nt_s, n = 0, \pm 1, \pm 2, ... \end{cases} \tag{2}$$

It is easy to notice that, if the input noise is white or has a time-correlation shorter that t_s, then the result for the noise transfer can be simply written as:

$$\overline{n_y^2(t)} = \int R_{xx}(\gamma) k_{w_{tt}}(\gamma) d\gamma = R_{xx}(0) \frac{1}{N} = \overline{n_x^2(t)} \frac{1}{N} \tag{3}$$

Remembering now that the input $(S/N)_x = A/\sqrt{\overline{n_x^2(t)}}$ we obtain:

$$(S/N)_y = \sqrt{N}(S/N)_x \tag{4}$$

Indeed, the signal to noise ratio has increased, due to the fact that each sample has the effect of averaging the input noise, thus resulting in au output noise which converges on its ensemble average (which is zero!).

More complex situations can be handled with DT, employing for example weights which scales as a power laws, *i.e.* $w_k = \alpha^k$, or as an exponential, *i.e.* $w_k = e^{kT_1/T_2}$. As a general rule, we can employ DT filters to mimic any continuous weighting function that we want.
Also for DT filters it is possible to work out results in the frequency domain. One thing that we need to remember is that a Dirac comb, *i.e.* a set of Dirac's delta functions equally spaced extending indefinitely in the time domain, has a Fourier transform which is a rescaled Dirac comb:

$$\mathcal{F}\left\{\sum_{-\infty}^{+\infty}\delta(t-nt_s)\right\} = \frac{1}{t_s}\sum_{-\infty}^{+\infty}\delta\left(f-\frac{n}{t_s}\right) \tag{5}$$

We can thus follow the general procedure in which we write the weighting function as an envelope $f(t)$ multiplied by a Dirac comb. The Fourier transform of the weighting function $W(t,f)$ will then be:

$$W(t,f) = \mathcal{F}\{f(\tau)\} * \frac{1}{t_s}\sum_{-\infty}^{+\infty}\delta\left(f-\frac{n}{t_s}\right) \tag{6}$$

Namely, an infinite set of shifted replicas of the Fourier transform of the envelope function centered at all the frequencies $f - n/t_s$. Similarly, the square modulus of the Fourier transform of the weighting function, which is the Fourier transform of the weighting function time-correlation, can be written following the same approach.

List of relevant exams

- Exam of 4$^{\text{th}}$ July 2016, exercise 2
- Exam of 23$^{\text{rd}}$ September 2016, exercise 2
- Exam of 23$^{\text{rd}}$ January 2018, exercise 2
- Exam of 16$^{\text{th}}$ January 2019, exercise 2
- Exam of 20$^{\text{th}}$ June 2019, exercise 2
- Exam of 22$^{\text{nd}}$ July 2021, exercise 2
- Exam of 20$^{\text{th}}$ January 2023, exercise 2
- Exam of 17$^{\text{th}}$ February 2023, exercise 2
- Exam of 6$^{\text{th}}$ September 2023, exercise 2