Electronics - Tutorage 7

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Amplitude modulation (AM)

Amplitude modulation is a technique which is vastly adopted when dealing with low frequency signals with superimposed low frequency noise. The typical case is the one of a constant or slowly variable signal buried in flicker noise.

To understand how we can exploit amplitude modulation to achieve the goal of increasing the (S/N) in the aforementioned situation we need to understand what's going on when we modulate a signal with another one. Let's start, first of all, by defining the term modulation: we say that a certain signal $x(t)$ is modulated by the signal $x_M(t)$ when we generate the signal $x(t) \cdot x_M(t)$. Note that, to perform such operation, a multiplier (*i.e.* a non-linear element) is needed. The modulation operation can be seen also as enveloping the low-frequency signal $x(t)$ on a carrier reference signal $x_M(t)$. Typical reference signals are periodic at a certain frequency f_M. From Fourier analysis this means that it is possible to decompose them in terms of $sin(k\omega_M t)$ and $cos(k\omega_M t)$. In that situation, moving to the frequency domain provides a more intuitive way of understanding the modulation operation.

Indeed, let's consider now our modulated signal $y(t) = x(t)x_M(t)$, where $x_M(t) = Acos(\omega_M t)$. Moving to the frequency domain we obtain:

$$Y(f) = X(f) * X_M(f)$$

But the Fourier transform of a cosine (and similarly of a sine) is given by two delta functions of amplitude $A/2$ and centered at $\pm f_M$. The modulation operation can thus be interpreted as a spectral shift, where the (bilateral) spectrum of the modulated signal is shifted such that there will be a replica of this spectrum (properly multiplied by the weight of each delta function) centered at each harmonic composing the modulation signal.

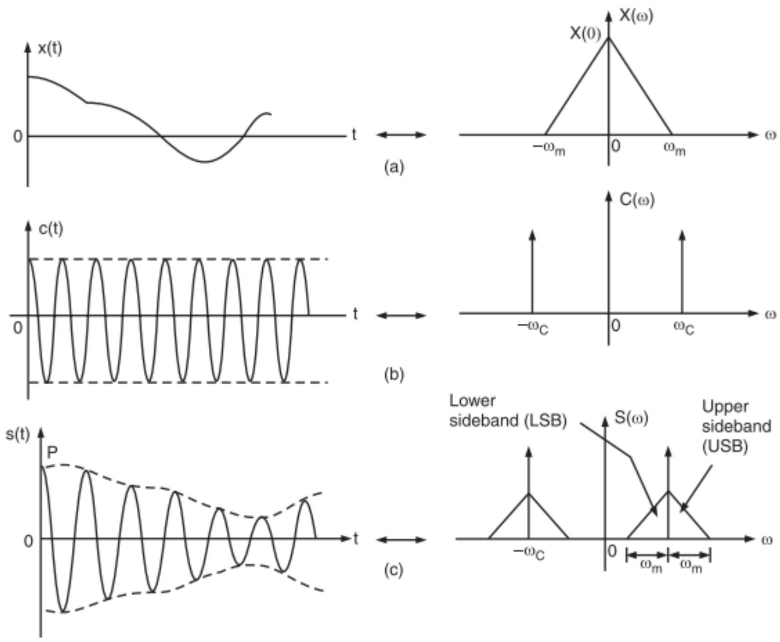

Figure 1: Graphic representation of the amplitude modulation operation both in the time (left) and in the frequency (right) domain.

Figure 1 illustrates what has been here described by showing the amplitude modulation operation on a general low-frequency signal.

When we use the term demodulation, we refer, instead, to a similar operation, which is able to bring back the spectrum into its original position around the frequency origin. To understand this, we can rely on the fact that, if we multiply two cosine (or sine) functions, oscillating at frequencies ω_M and ω_{DM}, we obtain cosine functions of frequencies $\omega_M \pm \omega_{DM}$. Thus, by simply multiplying again the modulated signal with another synchronous signal oscillating at the same frequency, we will be able to shift back the spectrum around the origin (with a loss of amplitude). From a frequency standpoint, we can interpret the demodulation stage as a further shift of the spectrum:

- modulating the spectrum we shifted it as to center it around the frequencies $\pm f_M$;
- demodulating it we further provide a shift of $\pm f_{DM}$ to each shifted replica;

If the modulation and demodulation frequencies are the same, two out of four replicas will be shifted back in the origin.
What we said can be demonstrated to hold also for stochastic signals, as noise. However, in general one will find that for the autocorrelation of the noise after the modulation stage it holds:

$$R_{yy}(\tau) = R_{xx}(\tau) \cdot K_{x_M x_M}(\tau)$$

Where with $K_{x_M x_M}(\tau)$ we indicated the autocorrelation of the modulating signal. From a frequency standpoint we can still interpret the modulation as a shift of the (bilateral) PSD. Indeed, if the modulation signal is a pure harmonic (sine or cosine), the Fourier transform of its autocorrelation will still be a set of two delta functions centered at $\pm$ the modulation frequency (weighting $A^2/4$ each one).
The operations of modulation/demodulation of deterministic and stochastic signals will be the starting point to understand the technique of synchronous detection exploited in lock-in amplifiers.

Lock-in amplifiers (LIA)

A lock-in amplifier acts on signals buried in noise in the following way:

- Modulate the low frequency signal buried in low frequency noise at a high frequency, where noise is much less. Note that this has to be done before the principal noise source
- Multiply the signal again, once noise has been summed, with a synchronous signal. This will modulate the noise to higher frequencies but bring back a part of the signal to the original base-band
- Apply a low-pass filter to remove the majority of modulated noise while maintaining the signal integrity

The basic idea is thus to exploit frequency shifting due to modulation as a way of separating the low-frequency noise from the low-frequency signal. Let's now do the example in which both the modulation and the demodulation signals are made of a cosine of frequency ω_M with amplitude of A and B. Our signal constant and of amplitude x while the noise has a bilateral PSD $S(f)$. After the modulation stage the signal is now $xAcos(\omega_M t)$. After the demodulation stage the signal becomes, instead, $xABcos^2(\omega_M t) = xAB\frac{1+cos(2\omega_M t)}{2}$. We assume that our LPF bandwidth is smaller than $2f_M$ such that only the constant component is left after filtering. Our noise PSD will be instead shifted in frequency as to be centered around $\pm f_M$. The two PSDs are also multiplied by $B^2/4$. When filtered, the output rms noise will be:

$$\overline{n_y^2} = \frac{B^2}{4} \cdot (2BW_n) \cdot (S(f_M) + s(-f_M)) = B^2 \cdot BW_n \cdot S(f_M)$$

Where we assumed the PSD to be almost constant on the LPF bandwidth and we recalled that the initial PSD is an even function of f. The resulting signal to noise ratio then becomes:

$$(S/N) = \frac{\frac{xAB}{2}}{\sqrt{B^2 \cdot BW_n \cdot S(f_M)}}$$

If we assume the signal to be modulated and demodulated by the same reference (*i.e.* $A = B$), we obtain:

$$(S/N) = \frac{xA}{\sqrt{4 \cdot BW_n \cdot S(f_M)}}$$

In more complex situations where the modulation and demodulation signals are not simple trigonometric functions we can still work out a general result by exploiting Fourier decomposition and adopting the intuitive approach of frequency shifting.

In general the signal at the end of the LIA stage can be computed following this procedure:

i Write the expression for the signal in the time domain as the product of the input signal $x(t)$ times the modulation and demodulation functions;

ii Exploit trigonometric expressions in order to write the signal as a sum of elements, each composed by the product of $x(t)$ times a sine/cosine function of given frequency (including $f = 0Hz$);

iii Keep only the term which enters the base band. Typically this is the term where $x(t)$ is multiplied by a constant.

Similarly, for the noise we can:

i Write down the weighting function $w_R(t)$ as the sum of sine/cosine terms of given frequencies, exploiting trigonometric formulas to convert products of sine/cosine functions into sums;

ii Calculate the time correlation of the weighting function $K_{w_R,w_R}(\tau)$. Recall that the time correlation of a constant terms C_1 is equal to C_1^2, while for a generic cosine function $B\cos(\omega_R t)$ the time correlation is $\frac{B^2}{2}\cos(\omega_R \tau)$;

iii Perform a Fourier transform on $K_{w_R,w_R}(\tau)$, obtaining $S_{K_w w}(f)$. The convolution of $S_{K_w w}(f)$ with the input noise PSD provides the output noise PSD. Note that the resulting convolution will be easy to calculate, as each cosine term in $K_{w_R,w_R}(\tau)$ will transform in a couple od Dirac's delta functions, *e.g.*, $\left(\frac{B^2}{2}\right)\cos(\omega_R \tau) \iff \frac{1}{2}\left(\frac{B^2}{2}\right)(\delta(f - f_R) + \delta(f + f_R))$;

iv Once the PSD at the input of the LIA's LPF is known, you can compute the output rms noise with the typical method employed during the course, approximating the PSD to be constant over the filter BW.

List of relevant exams

- Exam of 16$^{\text{th}}$ February 2015, exercise 2
- Exam of 10$^{\text{th}}$ July 2015, exercise 2
- Exam of 11$^{\text{th}}$ September 2015, exercise 2
- Exam of 5$^{\text{th}}$ September 2016, exercise 2
- Exam of 14$^{\text{th}}$ September 2017, exercise 2
- Exam of 10$^{\text{th}}$ September 2018, exercise 2
- Exam of 16$^{\text{th}}$ January 2020, exercise 2
- Exam of 17$^{\text{th}}$ July 2020, exercise 2
- Exam of 9$^{\text{th}}$ September 2021, exercise 2
- Exam of 8$^{\text{th}}$ September 2022, exercise 2
- Exam of 23$^{\text{rd}}$ June 2023, exercise 2